# The three-body problem

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Abstract. The three-body problem, which describes three masses interacting through Newtonian gravity without any restrictions imposed on the initial positions and velocities of these masses, has attracted the attention of many scientists for more than 300 years. In this paper, we present a review of the three-body problem in the context of both historical and modern developments. We describe the general and restricted (circular and elliptic) three-body problems, different analytical and numerical methods of finding solutions, methods for performing stability analysis, search for periodic orbits and resonances, and application of the results to some interesting astronomical and space dynamical settings. We also provide a brief presentation of the general and restricted relativistic three-body problems, and discuss their astronomical applications.

#### 1. Introduction

In the three-body problem, three bodies move in space under their mutual gravitational interactions as described by Newton's theory of gravity. Solutions of this problem require that future and past motions of the bodies be uniquely determined based solely on their present positions and velocities. In general, the motions of the bodies take place in three dimensions (3D), and there are no restrictions on their masses nor on the initial conditions. Thus, we refer to this as the *general three-body problem*. At first glance, the difficulty of the problem is not obvious, especially when considering that the two-body problem has well-known closed form solutions given in terms of elementary functions. Adding one extra body makes the problem too complicated to obtain similar types of solutions. In the past, many physicists, astronomers and mathematicians attempted unsuccessfully to find closed form solutions to the three-body problem. Such solutions do not exist because motions of the three bodies are in general unpredictable, which makes the three-body problem one of the most challenging problems in the history of science.

In celestial mechanics, the general three-body problem deals with gravitationally interacting astronomical bodies and intends to predict their motions. In our Solar System, the planets and asteroids move around the Sun, while the moons orbit their host planets, which in turn also move around the Sun. As typical examples of the three-body problem, we may consider the Sun-planet-planet, Sun-planet-moon, or Sun-planet-asteroid systems. The three-body problem representing the latter system can be significantly simplified because the mass of the asteroid is always negligible when compared to the mass of either the Sun or the planet, which means that the gravitational influence of the asteroid on the planet and the Sun can be omitted from the theory. If this condition is satisfied, then the general three-body problem becomes the restricted three-body problem, and there are two possibilities, namely, the two bodies with dominant masses move around their center of mass either along circular or elliptic orbits, which leads to the respective circular or elliptic restricted three-body problems. It is the circular restricted three-body problem (CR3BP) that has been the most extensively studied.

The three-body problem has been studied for over three hundred years. We wish to provide a brief historical overview of the most significant results that have contributed to the current developments in the field. We divide this overview into two parts: first, we highlight the time period from the publication of Newton's *Principia* in 1687 to the publication of Poincaré's *Les Méthodes Nouvelles de la Mécanique Céleste* published in 1892 (volume 1 and 2) and in 1899 (volume 3); in the second part we focus on developments that took place from 1900 to the present time.

# 1.1. From Newton to Poincaré

The three-body problem was formulated and studied by Newton [1687] in his *Principia*, where he considered the motion of the Earth and the Moon around the Sun. In this problem there are key elements such as the ratio of the Moon's mass,  $M_{\rm M}$ , to the

Earth's mass,  $M_{\rm E}$ , that is  $M_{\rm M}/M_{\rm E}=0.0123$ , which is a small but not negligible number. Another element is the tilt of the Moon's orbit to the Earth's orbit, about 5°. This makes the problem completely predictable despite the fact that the Earth's orbit around the Sun and the Moon's orbit around the Earth are almost perfect circles. Newton stated that the problem was very difficult to solve; however, he was able to obtain an approximate solution which agreed with observations to within 8%.

A special form of the general three-body problem was proposed by Euler [1767]. He considered three bodies of arbitrary (finite) masses and placed them along a straight line. Euler showed that the bodies would always stay on this line for suitable initial conditions, and that the line would rotate about the center of mass of the bodies, leading to periodic motions of all three bodies along ellipses. Around the same time, Lagrange [1772] found a second class of periodic orbits in the general three-body problem. He showed that if the bodies were positioned in such a way that they form a triangle of equal sides which would move along ellipses for certain initial conditions, preserving always their original configuration. The Euler and Lagrange solutions are now known as particular solutions to the general three-body problem, and they will be further discussed in Section 3.

Euler [1767] was the first to formulate the CR3BP in a rotating (or synodic) coordinate system. This was an important development in the study of the three-body problem. Lagrange also studied the CR3BP and demonstrated that there were five equilibrium points (now known as the Lagrange points) at which the gravitational forces of the bodies balanced out; the Trojan asteroids discovered in 1906 along Jupiter's orbit occupy space close to two of the Lagrange points.

Important contributions to the CR3BP were made by Jacobi [1836], who used the synodic (rotating) coordinate system originally introduced by Euler [1767] to demonstrate that there was an integral of motion, which is now named after him. The Jacobi integral was used by Hill [1877, 1878] to determine the motion of an asteroid in the three-body problem and to introduce the so-called zero velocity curves (ZVC), which establish regions in space where the bodies are allowed to move. Hill considered a special case of the CR3BP in which two masses were much smaller than the first one (the problem is now known as the Hill problem), and in this way he discovered a new class of periodic solutions. His main contribution was to present a new approach to solve the Sun-Earth-Moon three-body problem. After almost two hundred years since the original formulation of the problem by Newton [1687], Hill developed his lunar theory, which with some modifications made by Brown [1896], is still being used today in celestial mechanics [Gutzwiller, 1998].

In the second half of the nineteenth century, Poincaré studied and advanced the solution of the three-body problem. His monumental three-volume book *Les Méthodes Nouvelles de la Mécanique Céleste*, originally published in 1892-99, contains his most important contribution to the study of the CR3BP; the book was translated as *New Methods of Celestial Mechanics* and edited by L. D. Goroff in 1993. In this work, Poincaré developed a number of new qualitative methods to solve differential equations

and used them to identify and study possible periodic orbits, while demonstrating non-integrability of the system of equations describing the three-body problem. Poincaré's new methods allowed him to identify the unpredictability of the problem, and to discover the first manifestation of a new phenomenon, which is now commonly known as chaos. Poincaré submitted some of those results for the King Oscar II Birthday Competition and he was awarded the prize. The competition and Poincaré's other contributions are described by Barrow-Green [1997], Diacu and Holmes [1996] and Peterson [1993], and will not be repeated here.

# 1.2. From Poincaré to the present time

Poincaré [1892] studied Hill's problem and generalized Hill's definition of periodic orbits [Hill, 1877, 1878]. He was able to choose such initial conditions that resulted in periodic orbits in the special restricted three-body problem. Poincaré's work on the existence of periodic solutions in dynamical systems with one degree of freedom was extended by Bendixson [1901]. He formulated and proved a theorem that is now known as the Poincaré-Bendixson theorem, which gives a criterion for the existence of periodic solutions in such systems. The work of Poincaré led to systematic searches for periodic orbits in the three-body problem and their classification by Darwin [1897, 1909], Moulton et al. [1920] and Strömgren and Pedersen [1922], as well as extensive studies of stability of such orbits, which was initiated by Poincaré [1892] then continued by Levi-Civita [1901] and Lyapunov [1907], who significantly generalized Poincaré's approach and his results.

Further generalizations and extensions of Poincaré's ideas on the stability of periodic orbits were pursued by Birkhoff [1912]. He introduced the concept of recurrent motion, which requires a sufficiently long time so that the motion comes arbitrarily close to all its states of motion, and showed its relationship to orbital stability. In addition, Birkhoff [1913] proved the 'Last Geometric Theorem' formulated by Poincaré. The Poincaré-Birkhoff theorem states that there are infinitely many periodic orbits near any stable periodic orbit; it also implies the existence of quasi-periodic orbits. Birkhoff [1915] also developed a topological model for the restricted three-body problem in which an asteroid was confined to move inside an oval about one of the larger masses.

By searching for periodic and non-periodic solutions to the three-body problem, researchers realized that the differential equations describing the problem contain singularities at which the solutions are abruptly terminated. An example of such a termination are collisions between two or even all three bodies; thus, we have the so-called collision singularities. Once their presence is established, the next challenging task is to eliminate them by a process called regularization. An important work on singularities in the three-body problem was done by Painlevé [1896, 1897]. He determined that collisions were the only singularities and that collisions can be excluded by setting certain initial conditions for which the equations of motion of the three-body problem could be integrable using power series solutions. Painlevé did not find such

solutions but his work had a strong impact on other researchers in the field. The problem of the regularization of collisions between two bodies was investigated mainly by Levi-Civita [1903], Bisconcini [1906], and Sundman [1907, 1909, 1912]. They also studied the triple collisions and formulated theorems that allowed establishing conditions for such collisions [Siegel and Moser, 1991].

Realizing that the possibility of finding closed-form solutions to the three-body problem was unlikely, researchers considered finding infinite series solutions. Delaunay [1867] used canonical variables in the perturbation theory of the three-body problem. Following Delaunay's work, Lindstedt [1884] and Gyldén [1893] introduced infinite series and investigated the rates of their convergence, but they were not able to present a formal proof of the convergence. Poincaré [1892] clarified the concepts of convergence and divergence of the infinite series and exposed certain flaws in the methods developed by these authors. Painlevé [1896, 1897] also attempted to find solutions to the three-body problem given in terms of infinite series but failed as the others did before him. However, he clearly stated that such solutions were possible, at least in principle.

Indeed, Painlevé was right and Sundman [1912] found a complete solution to the three-body problem given in terms of a power series expansion. Unfortunately, Sundman's solution converges very slowly so that it cannot be used for any practical applications. An important question to ask is: Does the existence of Sundman's solution contradict the unpredictability of motions in the three-body problem postulated by Poincaré [1892]? Well, the answer is that it does not because trajectories of any of the three bodies cannot be determined directly from Sundman's solution, which means that the trajectories can still be fully unpredictable, as this is observed in numerical simulations (see Section 4, 5 and 8).

The three-body system considered here is a Hamiltonian system, which means that its total energy is conserved. Poincaré [1892] expressed the differential equations describing the three-body problem in the Hamiltonian form and discussed the integrals of motion [Barrow-Green, 1997]. In general, Hamiltonian systems can be divided into integrable and non-integrable. As shown above, even the systems that are in principle non-integrable may still have periodic solutions (orbits) depending on a set of initial conditions. Poincaré, Birkhoff and others considered quasi-periodic solutions (orbits) in such systems. The very fundamental question they tried to answer was: What happens to solutions of an integrable Hamiltonian system when the governing equations are slightly perturbed? The correct answer to this question was first given by Kolmogorov [1954] but without a formal proof, which was later supplied independently by Moser [1962] and Arnold [1963]; the formal theorem is now known as the Kolmogorov-Arnold-Moser (KAM) theorem. This theorem plays an important role in the three-body problem and in other Hamiltonian dynamical systems [Hilborn, 1994].

As mentioned above, periodic orbits in the three-body problem were discovered in the past by using analytical methods. By utilizing computer simulations, Hénon [1965, 1974] and Szebehely [1967] found many periodic orbits and classified them. Recent work by Šuvakov and Dmitrašinović [2013] shows that periodic orbits

are still being discovered. An interesting problem was investigated numerically by Szebehely and Peters [1967a,b], who considered three objects with their masses proportional to 3, 4, and 5 located at the vertices of a Pythagorean triangle with the sides equal to 3, 4, and 5 length units; they were able to show that two of these objects form a binary system but the third one was expelled at high speed from the system. More modern discoveries include a new 8-type periodic orbit [Moore, 1993, Chenciner and Montgomery, 2000, Šuvakov and Dmitrašinović, 2013] and 13 new periodic solutions for the general three-body problem with equal masses [Šuvakov and Dmitrašinović, 2013].

Modern applications of the three-body problem have been greatly extended to include the Earth, Moon, and artificial satellites, as well as the recently discovered distant extrasolar planetary systems (exoplanets). With the recent progress in detection techniques and instrumentation, there are over a thousand confirmed exoplanets orbiting single stars, either both or a single component within binary stars, and even triple stellar systems‡ along with thousands of exoplanet candidates identified by the NASA's Kepler space telescope§. A typical three-body system would be a single star hosting two exoplanets or a binary stellar system hosting one exoplanet; we discuss such systems in this paper.

The three-body problem is described in many celestial mechanics books such as Whittaker [1937], Wintner [1941], Pollard [1966], Danby [1988], Siegel and Moser [1991], Murray and Dermott [1999], Roy [2005] and Beutler [2005], and in review papers such as Holmes [1990], Szebehely [1997], Gutzwiller [1998] and Ito and Tanikawa [2007]. The books by Poincaré [1892], Szebehely [1967], Marchal [1990], Barrow-Green [1997] and Valtonen and Karttunen [2006] are devoted exclusively to the three-body problem and also discuss its many applications.

This paper is organized as follows: in Section 2 we formulate and discuss the general three-body problem; then we describe analytical and numerical studies of this problem in Sections 3 and 4, respectively; applications of the general three-body problem are presented and discussed in Section 5; the circular and elliptic restricted three-body are formulated and discussed in Sections 6 and 7, respectively; applications of the restricted three-body problem are given in Section 8; the relativistic three-body problem and its astrophysical applications are described in Section 9; and we end with a summary and concluding remarks in Section 10.

# 2. The general three-body problem

#### 2.1. Basic formulation

In the general three-body problem, three bodies of arbitrary masses move in a three dimensional (3D) space under their mutual gravitational interactions; however, if the

<sup>†</sup> http://exoplanet.eu/catalog/

<sup>§</sup> http://kepler.nasa.gov/mission/discoveries/candidates/

motions of the bodies are confined to one plane the problem is called the *planar general* three-body problem. Throughout this paper, we use Newton's theory of gravity to describe the gravitational interactions between the three bodies; the only exception is Section 9 in which we discuss the relativistic three-body problem.

To fully solve the general three-body problem, it is required for the past and future motions of the bodies to be uniquely determined by their present positions and velocities. Let us denote the three masses by  $M_i$ , where i=1, 2 and 3, and their positions with respect to the origin of an inertial Cartesian coordinate system by the vectors  $\vec{R}_i$ , and define the position of one body with respect to another by  $\vec{r}_{ij} = \vec{R}_j - \vec{R}_i$ , where  $\vec{r}_{ij} = -\vec{r}_{ji}$ , j=1, 2, 3 and  $i \neq j$ . With Newton's gravitational force being the only force acting upon the bodies, the resulting equations of motion are

$$M_i \frac{d^2 \vec{R}_i}{dt^2} = G \sum_{j=1}^3 \frac{M_i M_j}{r_{ij}^3} \vec{r}_{ij} , \qquad (1)$$

where G is the universal gravitational constant.

The above set of 3 mutually coupled, second-order, ordinary differential equations (ODEs) can be written explicitly in terms of the components of the vector  $\vec{R}_i$ , which means that there are 9 second-order ODEs; see Broucke and Lass [1973] for an interesting and useful form of the equations. Since these equations can be converted into sets of 2 first-order ODEs, there are actually 18 first-order ODEs to fully describe the general three-body problem. A standard mathematical procedure [Whittaker, 1937] allows solving such ODEs by quadratures if independent integrals of motion exist. Therefore, we shall now determine the number of integrals of motion for the general three-body problem [McCord et al., 1998].

# 2.2. Integrals of motion

Let us sum Eq. (1) over all three bodies and take into account the symmetry condition  $\vec{r}_{ij} = -\vec{r}_{ji}$ . The result is

$$\sum_{i=1}^{3} M_i \frac{d^2 \vec{R}_i}{dt^2} = 0 , \qquad (2)$$

and after integration, we obtain

$$\sum_{i=1}^{3} M_i \frac{d\vec{R}_i}{dt} = \vec{C}_1 , \qquad (3)$$

where  $\vec{C}_1 = \text{const.}$  One more integration yields

$$\sum_{i=1}^{3} M_i \vec{R}_i = \vec{C}_1 t + \vec{C}_2 , \qquad (4)$$

with  $\vec{C}_2 = \text{const.}$ 

Since the center of mass is defined as  $\vec{R}_{cm} = \sum_{i=1}^{3} M_i \vec{R}_i / \sum_{i=1}^{3} M_i$ , Eq. (4) determines its motion, while Eq. (3) shows that it moves with a constant velocity. The vectors  $\vec{C}_1$  and  $\vec{C}_2$  are the integrals of motion, thus we have 6 integrals of motion by taking the components of these vectors.

The conservation of angular momentum around the center of the inertial Cartesian coordinate system in the general three-body problem gives other integrals of motion. To show this, we take a vector product of  $\vec{R}_i$  with Eq. (2), and obtain

$$\sum_{i=1}^{3} M_i \vec{R}_i \times \frac{d^2 \vec{R}_i}{dt^2} = 0 , \qquad (5)$$

which after integration yields

$$\sum_{i=1}^{3} M_i \vec{R}_i \times \frac{d\vec{R}_i}{dt} = \vec{C}_3 , \qquad (6)$$

where  $\vec{C}_3 = \text{const.}$  Hence, there are 3 more integrals of motion.

An additional integral of motion is related to the conservation of the total energy of the system. With the kinetic energy  $E_{\rm kin}$  given by

$$E_{\rm kin} = \frac{1}{2} \sum_{i=1}^{3} M_i \frac{d\vec{R}_i}{dt} \cdot \frac{d\vec{R}_i}{dt} \,, \tag{7}$$

and the potential energy  $E_{\rm pot}$  defined as

$$E_{\text{pot}} = -\frac{G}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{M_i M_j}{r_{ij}} , \qquad (8)$$

where  $i \neq j$ , we have the total energy  $E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} \equiv C_4 = \text{const}$  to be also an integral of motion.

According to the above results, there are 10 (classical) integrals of motion in the general three-body problem, which means that the set of 18 first-order ODEs can be reduced to 8 first-order ODEs. Actually, there are 2 other integrals of motion, one related to the elimination of time and the other to the elimination of the so-called ascending node. The time can be eliminated by transforming one of the dependent variables as an independent variable [Szebehely, 1967, Barrow-Green, 1997]. Jacobi [1843] changed variables so that two bodies would orbit a third one, and showed that the difference in longitude between the ascending nodes is fixed at  $\pi$  radians, which becomes an integral of motion [Barrow-Green, 1997]. Having obtained the 12 integrals of motion, the system of 18 equations can be reduced to 6 equations. It has been proven that no other independent integrals of motion exist [Bruns, 1887, Poincaré, 1892, Whittaker, 1937, Szebehely, 1967, Valtonen and Karttunen, 2006].

Since  $E_{\rm kin} > 0$  and  $E_{\rm pot} < 0$  (see Eqs (7) (8), respectively),  $E_{\rm tot}$  or  $C_4$  can either be positive, negative or zero, and  $C_4$  can be used to classify motions of the general three-body problem [Roy, 2005]. In the case  $C_4 > 0$ , the three-body system must split, which means that one body is ejected while the remaining two bodies form a binary system. The special case of  $C_4 = 0$  is unlikely to take place in Nature; however, if it indeed occurred, it would result in one body escaping the system. Finally, the case  $C_4 < 0$  may lead to either escape or periodic orbits with the result depending on the value of the moment of inertia given by  $I = \sum_{i=1}^{3} M_i R_i^2$ ; for details see Roy [2005], Marchal [1990] and Valtonen and Karttunen [2006].

In addition to the integrals of motions, the virial theorem  $\langle E_{\rm kin} \rangle = -\langle E_{\rm pot} \rangle /2$ , where  $\langle E_{\rm kin} \rangle$  and  $\langle E_{\rm pot} \rangle$  is the time average kinetic and potential energy, respectively, can be used to determine the stability of the three-body problem and its statistical properties [Valtonen and Karttunen, 2006]. The system is unstable if its time average kinetic energy is more than two times higher than its time average potential energy.

# 2.3. Another formulation

The standard formulation of the general three-body problem presented in the previous section is often replaced by either the Hamiltonian formulation extensively used by Poincaré [1892], or by the variational formulation described in detail and used by Siegel and Moser [1991]. Here, we present only the Hamiltonian formulation.

Let us consider the so-called natural units and introduce G = 1 in Eqs (1) and (8). Moreover, we write  $\vec{R}_i = (R_{1i}, R_{2i}, R_{3i}) \equiv q_{ki}$ , where  $R_{1i}$ ,  $R_{2i}$  and  $R_{3i}$  are components of the vector  $\vec{R}_i$  in the inertial Cartesian coordinate system, and k = 1, 2 and 3. Using this notation, we define the momentum,  $p_{ki}$ , as

$$p_{ki} = M_i \frac{dq_{ki}}{dt} \,, \tag{9}$$

and the kinetic energy as

$$E_{\rm kin} = \sum_{k,i=1}^{3} \frac{p_{ki}^2}{2M_i} \,, \tag{10}$$

and introduce the Hamiltonian  $H = E_{\text{kin}} + E_{\text{pot}}$  that allows us to write the equations of motion in the following Hamiltonian form

$$\frac{dq_{ki}}{dt} = \frac{\partial H}{\partial p_{ki}} \quad \text{and} \quad \frac{dp_{ki}}{dt} = -\frac{\partial H}{\partial q_{ki}}, \tag{11}$$

which is again the set of 18 first-order ODEs equivalent to the set of 18 first-order ODEs given by Eq. (1).

The equations of motion describing the general three-body problem (either Eq. (1) or Eq. (11)) can be further reduced by considering the general Hill problem, in which  $M_1 >> M_2$  and  $M_1 >> M_3$ , however,  $M_3$  is not negligible when compared to  $M_2$ .

The governing equations describing this problem are derived by Szebehely [1967] and Simó and Stuchi [2000], who also discussed their applications to the Sun-Earth-Moon problem, known as the Hill lunar theory. In this paper, we limit our discussion of the Hill problem to its circular and elliptic versions of the restricted three-body problem, and describe them in Section 6.5 and 7.3, respectively.

## 2.4. General properties of solutions of ODEs

Solutions of the ODEs describing the general three-body problem can be represented by curves in 3D. In a special case, a solution can be a single point known also as a fixed point or as an equilibrium solution; a trajectory can either reach the fixed point or approach it asymptotically. Typically, periodic orbits are centered around the fixed points, which are called stable points or stable centers. Moreover, if a trajectory spirals toward a fixed point or moves away from it, the point is called a spiral sink or spiral source, respectively. There can also be a saddle point in which two trajectories approach the point and two leave it, but all other trajectories are kept away from it.

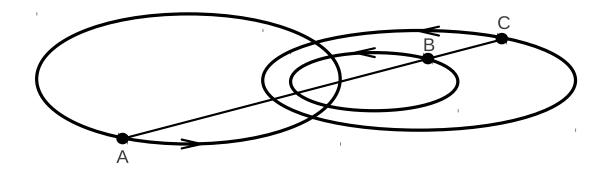
From a mathematical point of view, it is required that solutions to ODEs exist and that they are unique. The existence can be either global when a solution is defined for any time in the past, present and future, or local when it is defined only for a short period of time. The uniqueness of solution means that there is only one solution at each point. For given ODEs, the existence and uniqueness are typically stated by mathematical theorems. Let us consider the following first-order ODE y'(x) = f(y(x), x), where y' = dy/dx, and the intial condition is  $y(x_0) = y_0$ , with  $x_0$  being the initial value of  $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$ . Picard's (or Lipschitz and Cauchy's) existence theorem states that if f(y(x), x) is a Lipschitz continuous function in y and x, then there is  $\varepsilon > 0$  such that a unique solution y(x) exists on the interval  $[x_0 - \varepsilon, x_0 + \varepsilon]$ .

Another requirement is that the problem is well-defined, which means that a solution must be continuous with respect to the initial data. Being aware of the above mathematical requirements, Poincaré studied the problem of solving ODEs from another perspective by developing qualitative methods, which he described in the third volume of his book [Poincaré, 1892]; Poincaré's work was done mainly for the CR3BP, and we describe it in Section 6.

# 3. Analytical studies of the general three-body problem

#### 3.1. Euler and Lagrange periodic solutions

As already mentioned in Section 1, two different classes of periodic solutions to the general three-body problem were obtained by Euler [1767] and Lagrange [1772] who considered a linear and triangle configuration of the three bodies, respectively, and demonstrated that in both cases the bodies move along elliptic orbits. Since the masses of the three bodies are arbitrary (finite), the problem is considered here as the general three-body problem despite the limitation on the bodies mutual positions; because of



**Figure 1.** Illustration of the Euler solution with a line joining three masses  $M_1$ ,  $M_2$  and  $M_3$  located at the corresponding points A, B, and C.

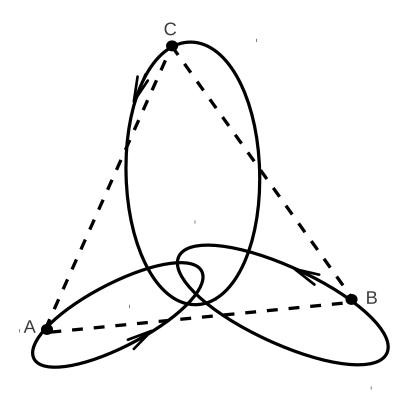
this limitation, the Euler and Lagrange solutions are called the particular solutions. The existence of the special configurations (known as central configurations in celestial mechanics) plays an important role in studies of orbital stability. Detailed derivations of the Euler and Lagrange solutions are given by Danby [1988].

In the Euler solution, the initial linear configuration of the three bodies is maintained (it becomes the central configuration), if the ratio AB/BC (see Fig. 1) has a certain value that depends on the masses, and if suitable initial conditions are specified. Euler proved that the line AC would rotate about the center of mass of the bodies leading to periodic motions of all three bodies along ellipses. Moreover, he also demonstrated that the ratio AB/BC would remain unchanged along AC during the motion of the bodies. Since the three bodies can be ordered in three different ways along the line, there are three solutions corresponding to the ordering of the bodies. However, it must be noted that the Euler solution is unstable against small displacements.

Now, in the Lagrange solution, the initial configuration is an equilateral triangle and the three bodies are located at its vertices. Lagrange proved that for suitable initial conditions, the initial configuration is maintained (becomes the central configuration) and that the orbits of the three bodies remain elliptical for the duration of the motion (see Fig. 2). Despite the fact that the central configuration is preserved, and the triangle changes its size and orientation as the bodies move, the triangle remains equilateral. Since the triangle can be oriented in two different ways, there are two solutions corresponding to the orientation of the triangle. Regions of stability and instability of the Lagrange solutions were identified by Mansilla [2006].

# 3.2. Other periodic solutions

Hill [1877, 1878] found periodic orbits in the problem that is now known as the Hill problem (see Section 1). Poincaré [1892] developed a method that allowed him to find periodic orbits in the CR3BP (see Section 6). However, the original version of his



**Figure 2.** Illustration of the Lagrange solution with an equilateral triangle joining three masses  $M_1$ ,  $M_2$  and  $M_3$  located at the corresponding points A, B, and C.

method can also be applied to the general three-body problem. We follow Poincaré [1892] and consider the autonomous Hamiltonian system given by

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}$$
 and  $\frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}$ , (12)

and  $F = F_0 + \mu F_1 + \mu F_2 + ...$ , where  $F_0 = F_0(x)$ , however,  $F_1$ ,  $F_2$ , ... are functions of both x and y and they are periodic (with a period of  $2\pi$ ) with respect to y. Moreover,  $\mu$  is a parameter that depends on mass, where i = 1, 2, and 3. If  $\mu = 0$ , then  $x_i = \text{constant}$  and  $y_i = \eta_i t + \omega_i$ , with  $\eta_i = \partial F_0 / \partial x_i$  and  $\omega_i$  being constants of integration, and a solution is periodic when the values of  $\eta_i$  are commensurable; actually, if  $\mu = 0$ , then there are infinitely many constants  $x_i$  that lead to periodic solutions [Barrow-Green, 1997].

Then Poincaré wanted to know whether the periodic solutions could be analytically continued when  $\mu$  remains small. The approach presented below follows Poincaré [1890], in which he assumed that the  $\mu \neq 0$  solutions at t=0 are:  $x_i=a_i+\delta a_i$  and  $y_i=\omega_i+\delta\omega_i$ , with  $a_i$  being constants. Now, for t=T, where T is the lowest common multiple of the  $2\pi/\eta_i$ , the solutions are:  $x_i=a_i+\delta a_i+\Delta a_i$  and  $y_i=\omega_i+\eta_i T+\delta\omega_i+\Delta\omega_i$ , which means

that they are periodic if all  $\Delta a_i = 0$  and all  $\Delta \omega_i = 0$ . Since only five equations given by Eq. (12) are independent (as F = const is their integral), Poincaré showed that the five equations could be satisfied if

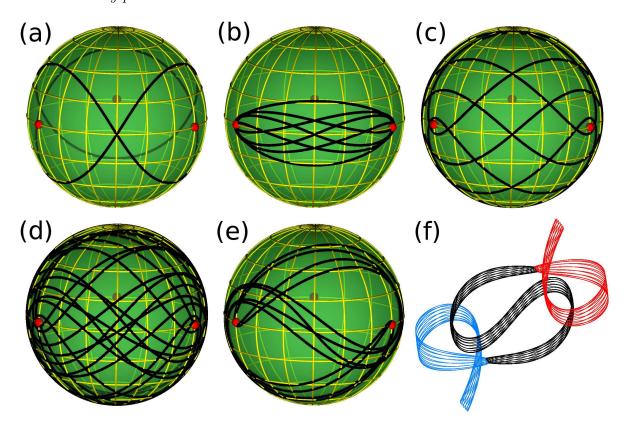
$$\frac{\partial \Psi}{\partial \omega_2} = \frac{\partial \Psi}{\partial \omega_3} = 0 , \qquad (13)$$

where  $\Psi$  is a periodic function with respect to  $\omega_1$  and  $\omega_2$ , and if the following additional conditions are also satisfied: the Hessian of  $\Psi$  with respect to  $\omega_1$  and  $\omega_2$  must be non-zero and the Hessian of  $F_0$  with respect to  $x_i^0$  must also be non-zero; the so-called Hessian condition is  $\text{Det}(\partial^2 \Psi/\partial \omega_1 \partial \omega_2) \neq 0$  and similar for  $F_0$ . Moreover, if  $\eta'_i = \eta_i(1+\varepsilon)$ , where  $\varepsilon$  is small, then there exists the following periodic solutions  $x_i = \phi_i(t, \mu, \varepsilon)$  and  $y_i = \phi'_i(t, \mu, \varepsilon)$  with period  $T' = T/(1+\varepsilon)$ .

Using this method, Poincaré established that there were periodic orbits for all sufficiently small values of  $\mu$ . Poincaré's approach was generalized first by the Poincaré-Bendixson theorem [Bendixson, 1901], and by the Poincaré-Birkhoff theorem [Birkhoff, 1912, 1913, 1915]. According to these theorems, there are periodic solutions even for non-integrable Hamiltonian systems. However, the theorems do not address a problem of what happens to solutions of an integrable Hamiltonian system when the governing equations are slightly perturbed? The answer to this question is given by the KAM theorem [Kolmogorov, 1954, Moser, 1962, Arnold, 1963] described in Section 3.3.

An important result concerning the existence of periodic solutions in the general three-body problem was obtained by Hadjidemetriou [1975a], who proved that any symmetric periodic orbit of the CR3BP could be continued analytically to a periodic orbit in the planar general three-body problem [Hadjidemetriou, 1975b]. Using this analytical result, families of periodic orbits in the planar general three-body problem were constructed [Bozis and Hadjidemetriou, 1976] and their stability was investigated [Hadjidemetriou and Christides, 1975]. Moreover, Katopodis [1979] demonstrated that it was possible to generalize Hadjidemetriou's result from the 3D CR3BP to the 3D general three-body problem.

Periodic orbits in the 3D general three-body problem are typically determined numerically [Hénon, 1965, 1974, Szebehely, 1967, Moore, 1993, Chenciner and Montgomery, 2000, Šuvakov and Dmitrašinović, 2013]. To make the computations as efficient as possible, the 3D general three-body problem was formulated suitably for numerical computations [Markellos, 1980] by using an idea of Hadjidemetriou and Christides [1975]. Periodic solutions with binary collisions in the general three-body problem were also determined analytically for different masses, and 8 different periodic orbits in a rotating frame of reference were found by Delibaltas [1983]. By starting with periodic orbits for the 3D CR3BP, families of periodic orbits in the 3D general three-body problem were numerically determined, and it was shown that the latter was not isolated but instead they form continuous mono-parametric families for given masses of the three bodies [Markellos, 1981]. Let us also point out that there are methods for constructing analytically periodic solutions to the general three-body problem by using computer experiments [Valtonen et al., 1989].



**Figure 3.** New periodic solutions are presented on the shape-space sphere with its back sides also visible. (a) The figure-8 orbit; (b) Butterfly orbit; (c) Moth orbit; (d) Yarn orbit; (e) Yin-yang orbit; (f) The yin-yang orbit in real space. Reproduced after Šuvakov and Dmitrašinović [2013]. Copyright 2013 Physical Review Letters.

More recently, new periodic solutions to the three-body problem were found using combinations of analytical and numerical methods. Specifically, Moore [1993] discovered, and Chenciner and Montgomery [2000] independently re-discovered, the so-called 8-type periodic solution for the general three-body problem with equal masses (see Fig. 3a), and formally proved its existence [Montgomery, 2001]. A search for periodic orbits in the vicinity of the 8-type periodic solution was performed numerically by Šuvakov [2013]. The KAM theorem was used by Simó [2002] to establish stability of this solution, and a rotating frame of reference was adapted to study properties of the solution [Broucke et al., 2006]. Moreover, new symmetric non-collision periodic solutions with some fixed winding numbers and masses were found by Zhang et al. [2004].

Currently, 13 new distinct periodic orbits were found by Šuvakov and Dmitrašinović [2013], who considered the planar general three-body problem with equal masses and zero angular momentum (see Fig. 3b,c,d,e,f). The authors presented a new classification of periodic solutions but admitted that their results could not be verified observationally because so far no astronomical systems with the considered properties of the three bodies are known.

## 3.3. The KAM theorem and stability of periodic solutions

Let us write Eq. (11) in the following general form

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \quad \text{and} \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} ,$$
(14)

where  $q = q_1, ..., q_n, p = p_1, ..., p_n$ , where t is time. It means that the equations describe the autonomous Hamiltonian system with n degrees of freedom. The Hamiltonian is  $H(p,q) = H_0(p) + \mu H_1(p,q) + ...$ , with H being periodic in q with period  $2\pi$ , and  $\mu$  is a small parameter.

For the unperturbed motion  $\mu = 0$ , the above equations reduce to

$$\frac{dq}{dt} = \frac{\partial H_0}{\partial p} = \Omega(p) \quad \text{and} \quad \frac{dp}{dt} = 0 ,$$
(15)

with  $\Omega = \Omega_1, ..., \Omega_n$ . Clearly, Eq. (12) can be integrated where the resulting trajectories are confined to a torus in the phase space, actually, there are invariant tori p = const; see Barrow-Green [1997] and Diacu and Holmes [1996] for details. For incommensurable frequencies the motion is quasi-periodic. Moreover, the system is nondegenerate because the Hessian determinant is not zero.

Now, the KAM theorem tells us that when the system is slightly perturbed most of the invariant tori are not destroyed but only slightly shifted in the phase space. This has important implications on stability of orbits in the general and restricted three-body problem. The proof of the KAM theorem by Moser [1962] and Arnold [1963] also demonstrated that convergent power series solutions exist for the three-body (as well as for the n-body) problem. The KAM theorem seems to be very useful for studying the global stability in the three-body problem [Robutel, 1993a, Montgomery, 2001, Simó, 2002]; however, some of its applications are limited only to small masses of the third body and as a result, different methods have been developed to deal with the general three-body problem [Robutel, 1993b].

## 3.4. Collision singularities and regularization

The Euler, Lagrange and other periodic solutions require suitable initial conditions, so an interesting question to ask is: What happens to the three bodies when the initial velocities are set equal to zero? Apparently, in this case all three bodies move toward their center of mass and undergo a triple collision at that point in finite time. At the point where the triple collision occurs, the solutions (if they are known) become abruptly terminated, which means that we have a triple collision singularity at that point. Similarly, we may have a double collision singularity when two bodies collide. In the following, we describe collision singularities in the general three-body problem and present a method called regularization to remove them.

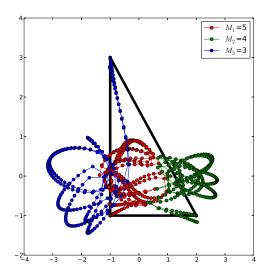
Based on the important work by Painlevé [1896, 1897], it was established that collisions between either two or three bodies are the only singularities in the three-body

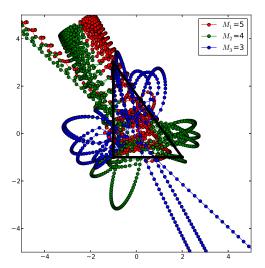
problem, and that such singularities could be removed by setting certain sets of initial conditions. The structure of phase space near the singularities changes and solutions (if they exist) do not end at the singularities but come close to them and show strange behaviours. The work of Painlevé on singularities resulting from collisions between two bodies was continued by Levi-Civita [1903] and Bisconcini [1906], with the former concentrated on the restricted three-body problem and the latter on the general three-body problem. They addressed an important problem of initial conditions that lead to collisions, and discussed regularization that allows extending possible solutions beyond a singularity. Bisconcini deduced two analytical relationships between initial conditions and proved that when the relationships were satisfied a collision did take place in a finite time.

The problem of triple collisions was investigated by Sundman [1907, 1909], who formulated and proved two important theorems. First he proved that the triple collisions would occur only if all the constants of angular momentum are zero at the same time. Second, he demonstrated that if all three bodies collide at one point in space they move in the plane of their common center of gravity, and as they approach the collision they also approach asymptotically one of the central configurations, namely, either Euler's collinear configuration or Lagrange's equilateral triangle. Proofs of Sundman's theorems and their applications are described in detail by Siegel and Moser [1991]. Sundman [1912] also studied binary collisions and used the results of those studies to find a complete solution to the general three-body problem (see Section 3.3). Moreover, Waldvogel [1980] considered the variational equations of the three-body problem with triple collisions and obtained solutions given in terms of hypergeometric functions, which are valid in the vicinity of the collision.

Regularization means that the motion is extended beyond a singularity, which occurs in a solution, through an elastic bounce and without any loss or gain of energy. Thus, it is important to know whether such a solution with a singularity can always be continued in a meaningful way? Siegel [1941] showed that in general this cannot be done by proving a theorem, which states that an analytical solution that goes through the triple collision cannot be found for practically all masses of the bodies. The work of Siegel was significantly extended by McGehee [1974], who obtained new transformations, now known as McGehee's transformations. He also introduced a new coordinate system that allowed him to magnify the triple collision singularity and inspect it more closely. What he found was a distorted sphere with four horns extending to infinity and he named this surface a collision manifold; for more details see Diacu and Holmes [1996]. There are two currently used regularization schemes, the so-called K-S regularization scheme developed by Kustaanheimo and Stiefel [1965], and the B-H regularization scheme developed by Burdet [1967] and Heggie [1973, 1976]. These two analytical schemes describe close two-body encounters and they are typically used to supplement numerical simulations [Valtonen and Karttunen, 2006].

In Section 1, we briefly discussed the numerical work by Szebehely and Peters [1967a,b], who considered three bodies with their masses proportional to 3, 4 and 5





**Figure 4.** Numerical solutions to the Pythagorean triangle problem show the evolution of the system for the first 10 orbits (left) and up to the ejection of the smallest mass from the system (right).

located at the vertices of a Pythagorean triangle with the sides equal to 3, 4 and 5 length units, and showed that two of these objects form a binary system but the third one is expelled from the system (see Fig. 4). Such a pure numerical result required an analytical verification and a formal mathematical proof that was given by McGehee [1974] for the three-body problem in which the bodies are restricted to move only along a fixed line. A more general proof for the planar three-body problem was given by Waldvogel [1976]. The main result of both proofs was that after a close triple encounter one of the bodies of the three-body problem would escape from the system. Moreover, sufficient conditions for the escape of one body from the general three-body system can also be specified [Standish, 1971, Anosova and Orlov, 1994] as well as the conditions that an ejected body would return to the system [Standish, 1972].

#### 3.5. Sundman's complete solutions

Painlevé [1896, 1897] formulated a conjecture that once collisions are excluded by choosing certain initial conditions, then the equations of motion of the general three-body problem could be integrable using power series solutions. Painlevé neither proved his conjecture nor found such solutions. Nevertheless, his results had strong impact on his contemporaries who searched extensively for the solutions but failed to find them. The only one who succeeded was Sundman [1907] who found a complete solution to the general three-body problem.

As already stated in Section 1, searches for power series solutions had been pursued before Painlevé formulated his conjecture. Different power series solutions were presented and their convergence rates were investigated [Delaunay, 1867, Lindstedt,

1884, Gyldén, 1893]. Some important points related to the convergence of those series were made [Poincaré, 1892] but he also did not succeed in finding the solution. A breakthrough occurred when Sundman [1907, 1909, 1912] studied double collision singularities. Let us now briefly describe Sundman's complete solutions to the general three-body problem by following Barrow-Green [1997].

Sundman considered a double collision singularity and showed that it could be removed by introducing a regularizing variable u defined in terms of time t as

$$u = \int_{t_0}^t \frac{dt}{t} \,, \tag{16}$$

where  $t_0$  corresponds to u = 0 for which the system is regular. With the singularity occurring at  $t = t_1$ , Sundman discovered that the coordinates of the three bodies could be expanded in powers of  $(t - t_1)^{1/2}$  and that he could use an analytical continuation to demonstrate that the expansion described correctly motions after the collision. The energy remained unchanged, the areal velocity stayed constant, and the solutions were valid for  $t > t_1$  and described motions correctly after each new double collision; note that the triple collisions were not allowed in this approach.

Since Sundman's regularization transformation was dependent on  $t_0$ , he introduced another variable, namely,  $dt = \Gamma d\omega$ , where  $\Gamma = (1 - e^{-r_0/l})(1 - e^{-r_1/l})(1 - e^{-r_2/l})$  and  $r_0$ ,  $r_1$  and  $r_2$  are the mutual distances, with the two greater of them being larger than l, which gives  $0 \le \Gamma \le 1$ , so there is a one-to-one correspondence between t and  $\omega$ .

If  $\omega^*$  is a real and finite value of  $\omega$ , then the coordinates of the three bodies, their mutual distances and time can be expanded in power series with respect to  $(\omega - \omega^*)$ , with the radius of convergence  $\Omega$  satisfying  $|\omega - \omega^*| \leq \Omega$ . Finally, Sundman defined

$$\tau = \frac{e^{\pi\omega/2\Omega} + 1}{e^{\pi\omega/2\Omega} + 1} \,, \tag{17}$$

and used

$$\omega = \frac{2\Omega}{\pi} \log \frac{1+\tau}{1-\tau} \,, \tag{18}$$

which allowed him to transform the  $\omega$ -plane into a circle of unit radius in the  $\tau$ -plane. This guarantees that the coordinates of the bodies, their mutual distances and time are analytic functions of  $\tau$  everywhere inside the circle. Moreover, the expansions are convergent and their different terms can be calculated once l and  $\Omega$  are specified. Thus, these are complete solutions to the general three-body problem.

Clearly, this is a remarkable result and this fact had been recognized by Sundman's contemporaries [Levi-Civita, 1918, Birkhoff, 1920]. However, there is a major problem with Sundman's solutions; namely, their convergence is extremely slow, actually so slow that it requires millions of terms to find the motion of one body for insignificantly short durations of time. Since the solutions are also not useful for numerical computations, owing to the round-off errors, they have no practical applications. Thus, we have a paradox: though we know the complete solutions, nothing is added to the previously

accumulated knowledge about the three-body problem! A more detailed discussion of Sundman's complete solutions and his other relevant results can be found in Siegel and Moser [1991] and Barrow-Green [1997].

## 3.6. Zero velocity hypersurfaces

As already mentioned in Section 1, Hill [1877, 1878] introduced the zero velocity curves (ZVC) for the restricted three-body problem; these curves are boundaries of regions in space where the bodies are allowed to move. Hill's results can be generalized to the 3D general three-body problem; however, in this case, regions of space where motions are disallowed are not curves but rather 4D surfaces, or hypersurfaces. The so-called zero velocity hypersurfaces (ZVH) can be determined using Sundman's inequality [Sergysels, 1986] and the resulting restrictions on the motions must obviously be consistent with the existing solutions to the problem [Marchal, 1990, Valtonen and Karttunen, 2006]. If the three bodies have masses  $M_1 > M_2 > M_3$ , and if the terms of the order  $(M_3/M_2)^2$  can be neglected, then the ZVH of the general three-body problem become the ZVC of the restricted three-body problem [Milani and Nobili, 1983].

# 4. Numerical solutions to the general three-body problem

Since the analytical solutions to the three-body problem obtained by Sundman [1907] converge extremely slowly, the modern practice of solving this problem involves numerical computations. In general, there are different numerical methods to solve the three-body problem, and determine periodic orbits, resonances and chaotic indicators. In the following, we present an overview of these methods, including numerical simulation packages such as *MERCURY*, *SWIFT*, and *HNBody* developed for astronomical applications [Chambers, 1999, Levison and Duncan, 1994, Rauch and Hamilton, 2002].

# 4.1. Numerical methods

One of the commonly used methods in determining numerical solutions is the application of symplectic integration. This method relies on the mathematical parametrization of the equations of motion into matrices and assumes a low occurrence of events where the orbital velocity could quickly become large. In celestial mechanics, this is an implicit assumption on the likely values of eccentricity for the system being considered. With low eccentricity, larger time steps can be implemented, which can significantly quicken the calculations. This is possible due to the symplectic nature of Hamiltonian dynamics where matrices are symmetric and easily invertible. Using this mathematical property, one can study nonlinear Hamiltonian dynamics with computationally fast mapping integration methods. Wisdom and Holman [1992] developed an algorithm that incorporates this idea and applied it to study the n-body problem.

In general, one can envision a basic mapping method such as Euler's or leapfrog's integration, implemented with symplectic properties to attain higher speed and accuracy over more computationally expensive methods such as the Runge-Kutta method [Press et al., 1992]. We now briefly outline the leapfrog method, and then demonstrate how it can be transformed into a symplectic method.

The leapfrog integration method applies to problems that can be expressed as a second order differential equation of the following form:

$$\ddot{x} = F(x). \tag{19}$$

Many celestial mechanics problems are of this form and could thus be easily solved using the leapfrog integration method, which is stable for oscillatory motion as long as the time step,  $\Delta t$ , is constant and less than  $2/\omega$ ; this parameter is related to the mean motion  $\omega = \sqrt{G(M+m)/a^3}$ , where a is a semi-major axis. For example, the numerical integration of Solar System dynamics has a maximum time step on the order of 0.0767 years. However, in practice, one will use a smaller time step to better sample Mercury's orbit. The choice of  $\Delta t$  will be at least an order of magnitude larger than other numerical methods and hence computationally quicker. The leapfrog method of integration utilizes half step evaluations of the following form:

$$x_{i} = x_{i-1} + v_{i-1/2}\Delta t,$$

$$a_{i} = F(x_{i}),$$

$$v_{i+1/2} = v_{i-1/2} + a_{i}\Delta t,$$
(20)

where  $a_i$  are coefficients determined by  $F(x_i)$ , and  $v_i$  is the velocity.

The main advantage of this method is its symplectic properties, which inherently conserve the orbital energy of the system. This is highly valued in the field of celestial mechanics. Also the leapfrog method is time-reversible, which means that one can integrate forward n steps to a future state and reverse the direction of integration to return to the initial state.

Other commonly used computational techniques can be categorized broadly as Runge-Kutta like methods. Different approaches to the general algorithm of Runge-Kutta have produced methods of determining the coefficients for the Runge-Kutta method to high order but with a decrease in computational speed. Such approaches include Bulirsch-Stoer, Dormand-Prince, and Störmer-Cowell techniques [Press et al., 1992], where variable time steps along with sophisticated predictor-correctors have been utilized to make them more adaptive and accurate when compared with symplectic methods. We show the basic approach of the Runge-Kutta-Fehlberg (RKF)

implementation following Press et al. [1992]. The basic generalization is

$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i,$$
where
$$k_1 = h f(t_n, y_n)$$

$$k_2 = h f(t_n + c_2 h, y_n + a_{21} k_1)$$
...
$$k_s = h f(t_n + c_s h, y_n + a_{s1} k_1 + a_{s2} k_2 + \dots + a_{s,s-1} k_{s-1}),$$
(21)

where  $y_n$  is a dependent variable,  $t_n$  is an independent variable, s is a number of stages, and the coefficients  $b_i$ ,  $c_i$  and  $a_{ij}$ , with i=1,2,...,s and  $1 \le j < i \le s$ , are determined from the so-called Butcher tableau, which give the relationships between these coefficients. The most common RKF implementation is RKF45, which describes a method with s=6. The adaptive portion of the method compares an order m method  $(y_{n+1})$  with an order m-1 approach  $(y_{n+1}^*)$ , where m=5 for RKF45. This comparison is used to determine the relative error between the orders, which will be a criterion to modify the step-size h. The error estimate  $e_{n+1}$  with the control scalar z is of the following form

$$e_{n+1} = y_{n+1} - y_{n+1}^* = \sum_{i=1}^s (b_i - b_i^*) k_i,$$

$$z = \left(\frac{\epsilon h}{2|e_{n+1}|}\right)^{\frac{1}{4}}.$$
(22)

In order to determine the value of z, one specifies the error control tolerance  $\epsilon$ . Then, the product of z with the step-size h is used as the new optimal step-size h' for the next iteration. When considering simulations where close encounters are highly probable, an adaptive step-size method is preferred. Otherwise large errors in the total energy can accrue and impact the reliability of the results.

A more classical method of integration uses the Taylor method, which relies on the knowledge of terms within a Taylor series. In general, the optimal number of terms in the Taylor series may not be known, thus optimization packages such as TAYLOR uses a text input of the ODEs to be solved and provides the user with an optimal time-stepper that will minimize error [Jorba and Zou, 2005]. This method can be comparable in terms of speed and accuracy with the other methods presented; however implicit assumptions are made that would need to be evaluated before a specific application.

# 4.2. Numerical search for periodic orbits and resonances

Periodic orbits have been determined analytically for special cases with the Euler and Lagrange solutions being the most notable. However, other analytical solutions have also been found in other periodic families by using numerical methods (see Šuvakov [2013] for details). Studies of periodic orbits are related to the topic of resonance. Mardling [2008] and Murray and Dermott [1999] give detailed descriptions of the perturbation theory involved in determining the locations of resonance based on approximations to a harmonic oscillator. Using the perturbation theory, appropriate initial guesses can be made as to the locations of periodic orbits and resonances within a parameter space of semi-major axis and eccentricity. Such guesses are made using the ratio of Kepler's harmonic law between two orbiting bodies.

The nominal resonance location,  $R_a$ , can be defined in terms of two integers (k, l), where l is the order the resonance to form a ratio of k/(k+l). For internal resonances, the third body orbits between the primary and secondary, and the relation is given by

$$R_a = \left(\frac{k}{k+l}\right)^{2/3} \left(\frac{m_p}{m_p + m_s}\right) a \tag{23}$$

where a,  $m_p$ , and  $m_s$  denote the semi-major axis of the secondary, mass of the primary, and mass of the secondary, respectively. There is a similar definition for the external resonances.

Knowing this relation, one can begin to search for resonances that lead to periodic orbits using the general form of the resonant argument as

$$\phi = j_1 \lambda' + j_2 \lambda + j_3 \varpi' + j_4 \varpi + j_5 \Omega' + j_6 \Omega , \qquad (24)$$

with conditions on the  $j_n$  integer coefficients using the d'Almebert rules [Hamilton, 1994], which require a zero sum of the coefficients and an even sum for the  $j_5$  and  $j_6$  coefficients due to symmetry conditions. Also, the  $\lambda'$ ,  $\varpi'$ ,  $\Omega'$  represent the mean longitude, longitude of pericenter, and ascending node of the perturber. Accounting for all these conditions, a numerical search for resonances can be performed in a tractable way using the period ratio P'/P as an initial guess.

Eq. 24 can be characterized in terms of mean motion and secular components. When considering mean motion resonances, the  $j_1 - j_4$  coefficients are included such that they obey the zero sum condition, and the remaining terms are assumed to be zero. The search for secular resonances usually consider only the condition where  $j_3 = -j_4 = 1$  and all other coefficients are zero. The differences between these types of resonances manifests in the changes of the Keplerian orbital elements, which define an orbit. Further details on these changes can be found in Murray and Dermott [1999], and their applicability of study within the Solar System.

Another method of searching for periodic orbits includes the characterization of chaos in the three-body problem. Chaotic regions can alter the periodicity of the orbit, as well as the orientation of the orbit relative to some reference direction (i.e., precession of Eulerian angles). Goździewski et al. [2013] and Migaszewski et al. [2012] demonstrated this principle with a chaos indicator (for details, see Section 4.5).

# 4.3. Maximum Lyapunov exponent

Time evolution of dynamical systems can be characterized through the use of the method of Lyapunov exponents [Lyapunov, 1907]. First applications of these exponents to the general three-body problem were made by Benettin et al. [1976, 1980]; specific applications to the restricted three-body problem were first made by Jefferys and Yi [1983]. The method of Lyapunov exponents utilizes a description of two nearby trajectories in phase space for a third mass within the phase space. The mathematical description follows a power law (usually linear) for stable orbits, whereas an exponential law is used for unstable orbits, with the Lyapunov exponent being a multiplicative power of such laws (i.e.,  $x^{\pm \lambda t}$ ,  $e^{\pm \lambda t}$ ).

Most generally, the rate of divergence resulting from these laws is divided into 3 regimes. A positive Lyapnuov exponent  $\lambda$  denotes that a trajectory, corresponding to one degree of freedom, is diverging from the neighboring orbit, with the rate of divergence being characterized by a time series  $\lambda(t)$ . A negative Lyapunov exponent implies that a trajectory is converging onto a stable manifold and can also be further described by a time series. The third case considers a Lyapunov exponent that is exactly equal to zero, which means that the two trajectories are parallel to each other and by induction will remain so unless another force arises to disrupt the system. The mathematical definition of the Lyapunov exponent [Hilborn, 1994, Musielak and Musielak, 2009] is

$$\lambda = \lim_{t \to \infty} \frac{1}{t - t_0} \ln \left( \frac{\delta(t)}{\delta(t_0)} \right) . \tag{25}$$

Using the above basic description, one can calculate the spectrum or set of Lyapunov exponents for a given system with n degrees of freedom in a 2n dimensional phase space. Within the CR3BP, the system can be rotated with respect to the motion of the two large masses, so n=3 degrees of freedom exist within a 6 dimensional phase space. Since in this example, the system is Hamiltonian, the trace of the Jacobian J must be zero, which presents a symmetry among the Lyapunov exponents, with 3 positive and 3 negative exponents existing. There are other examples for which such symmetry does not exist, and only 2 regimes, chaotic or dissipative, can be identified. The sum of the exponents (Tr J) can then be positive or negative, and this sum determines whether a system is chaotic or dissipative, respectively.

The Jacobian for the ith mass is given by

$$J_{i} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial \xi_{i}}{\partial \xi_{i}} & \frac{\partial \xi_{i}}{\partial \eta_{i}} & \frac{\partial \xi_{i}}{\partial \zeta_{i}} & 0 & 0 & 0 \\ \frac{\partial \tilde{\eta}_{i}}{\partial \xi_{i}} & \frac{\partial \tilde{\eta}_{i}}{\partial \eta_{i}} & \frac{\partial \tilde{\eta}_{i}}{\partial \zeta_{i}} & 0 & 0 & 0 \\ \frac{\partial \xi_{i}}{\partial \xi_{i}} & \frac{\partial \tilde{\eta}_{i}}{\partial \eta_{i}} & \frac{\partial \zeta_{i}}{\partial \zeta_{i}} & 0 & 0 & 0 \\ \frac{\partial \xi_{i}}{\partial \xi_{i}} & \frac{\partial \xi_{i}}{\partial \eta_{i}} & \frac{\partial \zeta_{i}}{\partial \zeta_{i}} & 0 & 0 & 0 \end{pmatrix},$$

$$(26)$$

where the coordinates  $(\xi, \eta, \zeta)$  denote an inertial reference frame. The accelerations are given by Newton's Universal Law of Gravitation, i.e.,  $\mathbf{a}_i = -\sum_{i\neq j} \frac{GM}{r_{ij}^3} \hat{\mathbf{r}}_{ij}$  and  $r_{ij}^2 = (\xi_i - \xi_j)^2 + (\eta_i - \eta_j)^2 + (\zeta_i - \zeta_j)^2$ .

# 4.4. Fast Lyapunov Indicator

Since the calculation of the Jacobian involves extra computational time, other methods have been developed to determine more efficiently the onset of chaos. We discuss two such methods: the Fast Lyapunov Indicator (FLI) and the Mean Exponential Growth Factor of Nearby Orbits (MEGNO); the latter is discussed in the next section. Using these methods chaos can be detected within 100,000 year timescales; note that previous methods required much longer timescales.

The FLI is a method, which identifies two shortcomings of the previous approach of using the Lyapunov exponents, namely, normalization and dimensionality [Froeschlé et al., 1997]. The problem of dimensionality is related to the computational time used to follow non-chaotic vectors within the phase space. Thus, the supremum of the tangent vectors is used to determine the chaos indicator, with other vectors being neglected. This is shown by the following standard formula:

$$FLI(t) = \sup_{i} ||v_j(t)|| , \qquad (27)$$

where  $j = 1 \dots m$  of a m-dimensional basis.

The slope of the time series of FLI are used as indicators of chaos. If the slope is steep and positive, then chaos is inferred; this is analogous to a positive Lyapunov exponent. The slope may also remain flat, neither increasing or decreasing, which is the stable case. Lastly, there is the possibility for a negative slope, but this is not generally realized in celestial mechanics as the systems under study are Hamiltonian and dissipation would imply a loss of energy.

## 4.5. Mean exponential growth factor of nearby orbits

An alternate technique for identification of chaos is the use of the MEGNO; this method is based on the maximum Lyapunov exponent but in a slightly different manner. The method was developed by [Cincotta and Simó, 1999, 2000] as a general tool in probing chaos of Hamiltonian systems. Since that time, it has been used to probe the dynamics of many different astrophysical systems [Goździewski et al., 2001, 2002, Goździewski, 2002, Laughlin et al., 2002, Pilat-Lohinger et al., 2003]. For examples using the MEGNO method and other chaos indicator methods, we recommend Pavlov and Maciejewski [2003], Kotoulas and Voyatzis [2004], Barnes and Greenberg [2006], Funk et al. [2009], Dvorak et al. [2010], Hinse et al. [2010], Mestre et al. [2011], and Satyal et al. [2013].

The MEGNO method employs the integral form of determining the maximum Lyapunov exponent. In this sense, it can be interpreted as a measurement of the timeaverage mean value of the finite time Lyapunov exponent with the following definition [Goździewski et al., 2001, Hinse et al., 2010]

$$Y(t) = \frac{2}{t} \int_0^t \frac{\dot{\delta}(s)}{\delta(s)} s \, ds, \tag{28}$$

along with its time-averaged mean value

$$\langle Y \rangle (t) = \frac{1}{t} \int_0^t Y(s) ds.$$
 (29)

The advantage of this representation is that  $\langle Y \rangle$  (t) converges faster to its limit value even with the cost of two additional equations of motion [Goździewski et al., 2001]. This is due to the fact that the time-weighting factor amplifies the presence of stochastic behavior, which allows the early detection (i.e., shorter integration timescales) of chaos. The additional differential equations are simply the differential forms of Eqs. (28) and (29) given respectively as

$$\dot{x} = \frac{\dot{\delta}}{\delta}t$$
 and  $\dot{w} = 2\frac{x}{t}$ , (30)

where Y(t) = 2x(t)/t and  $\langle Y \rangle(t) = w(t)/t$ , with all the variables having their usual meaning.

# 5. Applications of the general three-body problem

The general three-body problem has been considered in many astronomical and spaceflight settings, including galactic dynamics, stellar formation, and also the determination of trajectories for spacecraft missions, such as manned, satellite, and robotic landers.

## 5.1. Astronomical settings

Exploring the general three body problem within an astronomical setting can encompass a broad scale. Here we discuss only the most basic cases. In order for a three-body problem to be considered general, all three masses must interact with each other with no limitations on the eccentricity of their orbits. One example is the case of triple stellar systems that are not hierarchical. The basic dynamics of these systems can only be investigated numerically because a variable timestep method of integration (see Section 4) is required due to Kepler's Second Law. The masses in such settings are likely to undergo several close approaches and great care must be taken during the point of close encounter between any of the masses.

As previously discussed (see Section 3.4), the Pythagorean problem exemplifies the difficulty in obtaining meaningful results in the face of large sources of numerical error. In terms of stellar dynamics, Reipurth and Mikkola [2012] investigated this setup. Examples include a finite sized cloud core around the three masses, which presents an even more intricate dynamical problem as the masses of the three bodies increase with

time depending on the interaction with the cloud core. These types of investigations provide an insight into the formation of wide binary stars (> 100 AU separations) due to dynamical instabilities in the triple systems.

Conversely the general three body problem has been used to explain the formation of close binaries (< 100 AU separations). Results of numerical evaluation of orbits along with secular perturbations have explained the occurrence of close binaries [Fabrycky and Tremaine, 2007], who provided a specific mechanism for the formation of such systems through the Lidov-Kozai cycles [Lidov, 1962, Kozai, 1962]; within these cycles large variations of eccentricity and inclination occur. The idea that stellar binary system may host planets has been explored in great detail [Gonczi and Froeschle, 1981, Rabl and Dvorak, 1988, Dvorak et al., 1989, Holman and Wiegert, 1999, Pilat-Lohinger and Dvorak, 2002, Pilat-Lohinger et al., 2003, Dvorak et al., 2004, Musielak et al., 2005, Pilat-Lohinger and Dvorak, 2008, Haghighipour et al., 2010, Cuntz, 2014], with the additional observational verification by the NASA Kepler mission (see Section 1). The consideration of binaries hosting planets will be addressed in more detail in Sections 8 within different regimes of approximation.

# 5.2. Exoplanets

Recently, using the *Kepler* data, over 700 exoplanet candidates have obtained the status of exoplanets based on a statistical method to identify the likelihood of false positive candidates among the sample [Lissauer et al., 2014, Rowe et al., 2014]. This portends to the general three body problem as each considered case includes the possible interaction of three masses. This is due to the possible geometric configurations in the photometric method. Some examples include the presence of third light contamination, background eclipsing binaries, and stellar triple systems. Prior to this major identification of exoplanets, a method called transit timing variations (TTVs) has been used to identify systems with multiple planets (Kepler-11, Kepler-36, Kepler-9); the method determines how three-body interactions would affect the timing of transits relative to a linear prediction [Lissauer et al., 2011, Carter et al., 2012, Holman et al., 2010]. We present the basics of TTVs and defer the reader to a more in depth description by Winn [2011].

The method of TTVs relies on the observable deviation from Keplerian dynamics. The deviation reveals the existence of additional bodies, which in principle is a similar method that was used in the discovery of Neptune through variations in the orbit of Uranus [Adams, 1846, Airy, 1846, Challis, 1846, Galle, 1846]. Using the times of transit and the planetary period of a transiting planet, a fitting is made to a linear function to predict the subsequent eclipses  $(t_n = t_o + nP)$  of that planet. When the planet eclipses a host star early or late relative to the predicted time of the fitted function, a TTV is observed. The power of this method was illustrated by Nesvorný et al. [2012] with the detection of a non-transiting planet (KOI-872c) inferred by observations of the seen transiting planet (KOI-872b). The vetting of this planet underwent intense scrutiny from the perspective of the three-body problem as only particular solutions allowed for

the determination of the planet's mass as well as keeping the perturbing planet stable.

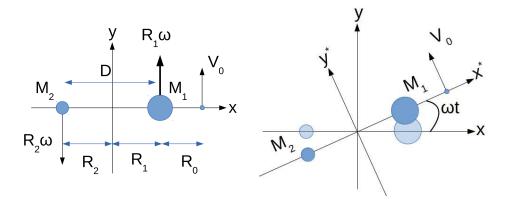
Many other discoveries have been made detecting multiple planet systems, including KOI-142 which is another discovery of a planet pair based on their mutual interactions [Nesvorný et al., 2013]. Much previous work was related to predicting the degree to which discoveries could be made based on the possible perturbations, and involved determining the sensitivity required to perform observations [Miralda-Escudé, 2002, Holman and Murray, 2005, Agol et al., 2005]. TTVs along with transit duration variations (TDVs) have even been suggested as possible methods to detect the presence of extrasolar moons [Kipping, 2009], although none so far have yet to be observed. Additional discussion on the requirements for the detection of extrasolar moons is presented in §8.5.

To account for exoplanets and exomoons in extra-solar planetary systems with the number of objects higher than three, additional new tools, such as *TTVfast* and *photodynam*, have been developed with full n-body capabilities built-in [Deck et al., 2014, Carter et al., 2011, Pál, 2012].

# 5.3. Spacecraft trajectories

Spaceflight applications involve the determination of spacecraft trajectories for satellite and lander space missions. NASA's Apollo 8 - 10 missions to the Moon required accurate numerical solutions to the three-body problem. NASA also launched several discovery missions, with the most notable being the Hubble Space Telescope, CHANDRA X-Ray Observatory, SPITZER Space Telescope, Kepler Space Telescope, and the upcoming James Webb Space Telescope. Each of these missions involve different solutions to three-body problem and all but Kepler orbit the Earth. One of the most interesting satellites will be the Transiting Exoplanet Survey Satellite (TESS||) as it will be in a high-Earth orbit and in a 2:1 resonance with the Moon. For these applications, it is required to consider the motions of the terrestrial objects relative to each other, including the motions of the Moon and Mars. The orbits of both of these objects can be affected by the perturbations from other planets.

More basic applications of the general three body problem involve artificial satellites commonly used for communication and/or military purposes. These satellites are generally considered within the realm of a restricted three-body problem (see Sections 6, 7 and 8). A restricted three body problem simplifies the solution because one of the masses is much less than the others and has negligible gravitational influence. Thus, the much larger masses have approximately Keplerian orbits. This is the case for the Earth-Moon-satellite system, and it is discussed in detail in Section 8.



**Figure 5.** Non-rotating and rotating coordinate systems used to describe the CR3BP. Reproduced after Eberle et al. [2008]. Copyright 2008 Astronomy & Astrophysics.

## 6. The circular restricted three-body problem

## 6.1. Governing equations

Based on the criteria stated in Section 1, the circular restricted three-body problem (CR3BP) requires that the two objects have their masses significantly larger than the third one  $(M_1 >> M_3 \text{ and } M_2 >> M_3)$ , and that the motions of  $M_1$  and  $M_2$  are limited to circular orbits around their center of mass. In the literature devoted to the CR3BP,  $M_1$  and  $M_2$  are typically referred to as the primaries [Szebehely, 1967], but in stellar dynamics  $M_1$  and  $M_2$  are called the primary and secondary, respectively, with the assumption that  $M_1 \geq M_2$ . In this paper, we use the former nomenclature when a clear distinction between  $M_1$  and  $M_2$  is not necessary; however, we use the second nomenclature when a distinction between  $M_1$  and  $M_2$  becomes important in describing a specific astronomical application.

Now, if the third mass moves in 3D, this case is called the 3D CR3BP; if it moves in the same plane as the primaries, we call this case the planar CR3BP. Since the gravitational influence of the third body on the primaries is negligible, the orbits of the primaries are described by the two-body problem, whose solutions are well-known. Once the solutions for the primaries are known, they can be used to determine the motion of the third body resulting from the gravitational field of the primaries. From a mathematical point of view, the problem becomes simpler than the *general* three-body problem. Nevertheless, for some sets of initial conditions the resulting motions of the third body remain unpredictable.

Let us choose the origin of a non-rotating coordinate system (see Fig. 5) to be at the center of mass of the primaries, so we can write  $M_1\vec{R}_1 = M_2\vec{R}_2$ , where  $M_1$  and  $M_2$  are the masses of the primaries with  $\vec{R}_1$  and  $\vec{R}_2$  denoting their position vectors with respect to the origin of the coordinate system. To describe the problem mathematically, we follow Szebehely [1967], Danby [1988] and Roy [2005].

The components of the position vectors are:  $\vec{R}_1 = (X_1, Y_1, Z_1), \vec{R}_2 = (X_2, Y_2, Z_2)$ 

and  $\vec{R}_3 = (X_3, Y_3, Z_3)$ , and they are used to express  $\vec{r}_{31} = \vec{R}_1 - \vec{R}_3 \equiv \vec{r}_1$  and  $\vec{r}_{32} = \vec{R}_2 - \vec{R}_3 \equiv \vec{r}_2$  in terms of their components. The equations of motion for the CR3BP can be written as

$$\frac{d^2\vec{R}_i}{dt^2} = G\sum_{j=1}^2 \frac{M_j}{r_j^3} (\vec{R}_j - \vec{R}_3) , \qquad (31)$$

where  $r_j = [(X_j - X_3)^2 + (Y_j - Y_3)^2 + (Z_j - Z_3)^2]^{1/2}$ .

We define  $M = M_1 + M_2$ ,  $\mu = M_2/M$  and  $\alpha = 1 - \mu$ , which give  $M_1 = \alpha M$  and  $M_2 = \mu M$ . With the gravitational force being equal to the centripetal force, we have

$$V_i^2 = R_j \frac{GM_{3-j}}{D^2} = R_j^2 \omega^2 , \qquad (32)$$

where  $D = R_1 + R_2$ ,  $\omega$  is the angular frequency or mean motion, and j = 1 and 2. Since  $M_{3-j} = MR_j/D$ , Kepler's third law  $\omega^2 = GM/D^3$  is automatically obtained. Moreover, we also have  $R_1 = \mu D$  and  $R_2 = \alpha D$ .

Now, once the large two masses are in circular orbits, we write:  $X_1(t) = \mu D \cos \omega t$ ,  $Y_1(t) = \mu D \sin \omega t$ ,  $X_2(t) = -\alpha D \cos \omega t$  and  $Y_2(t) = -\alpha D \sin \omega t$ , with  $Z_1 = Z_2 = 0$ , if the orbits are in the same plane.

Introducing  $X_3 = Dx$ ,  $Y_3 = Dy$  and  $Z_3 = Dz$ , and allowing the third body to move in 3D, we use Eq. (31) to derive the following set of equations of motion

$$\ddot{x} = -\frac{\alpha}{r_1^3} (x - \mu \cos \tau) - \frac{\mu}{r_2^3} (x + \alpha \cos \tau) , \qquad (33)$$

$$\ddot{y} = -\frac{\alpha}{r_1^3} (y - \mu \sin \tau) - \frac{\mu}{r_2^3} (y + \alpha \sin \tau) , \qquad (34)$$

and

$$\ddot{z} = -\left(\frac{\alpha}{r_1^3} + \frac{\mu}{r_2^3}\right) z , \qquad (35)$$

where  $\tau = \omega t$ , and  $\ddot{x}$ ,  $\ddot{y}$  and  $\ddot{z}$  represent the second-derivative of x, y and z with respect to  $\tau$ , respectively. This set of equations describes the CR3BP in the non-rotating coordinate system.

We write the above set of equations in the rotating (synodic) coordinate system by using the following relationships between the coordinates x, y and z in the inertial system, and the coordinates  $x^*$ ,  $y^*$  and  $z^*$  in the synodic system:  $x = x^* \cos \tau - y^* \sin \tau$ ,  $y = x^* \sin \tau + y^* \cos \tau$  and  $z = z^*$ . Therefore, the set of equations describing the CR3BP in the synodic coordinate system can be written as [Eberle, 2010]

$$\ddot{x}^* - 2\dot{y}^* = x^* - \frac{\alpha}{r_1^3} (x^* - \mu) - \frac{\mu}{r_2^3} (x^* + \alpha) , \qquad (36)$$

$$\ddot{y}^* + 2\dot{x}^* = \left(1 - \frac{\alpha}{r_1^3} - \frac{\mu}{r_2^3}\right) y^* , \qquad (37)$$

and

$$\ddot{z}^* = -\left(\frac{\alpha}{r_1^3} + \frac{\mu}{r_2^3}\right) z^* , \qquad (38)$$

where  $r_1 = D[(x^* - \mu)^2 + (y^*)^2 + (z^*)^2]^{1/2}$ , and  $r_2 = D[(x^* + \alpha)^2 + (y^*)^2 + (z^*)^2]^{1/2}$ .

## 6.2. Lagrange points

Lagrange [1772] found interesting solutions to the CR3BP that describe equilibrium positions of the third body when all net forces acting on it are zero. In this case, Eqs (36), (37), and (38) reduce to

$$x^* - \frac{\alpha}{r_1^3}(x^* - \mu) - \frac{\mu}{r_2^3}(x^* + \alpha) = 0 , \qquad (39)$$

$$y^* \left( 1 - \frac{\alpha}{r_1^3} - \frac{\mu}{r_2^3} \right) = 0 , \qquad (40)$$

and

$$z^* \left( \frac{\alpha}{r_1^3} + \frac{\mu}{r_2^3} \right) = 0 \ . \tag{41}$$

Taking  $z^* \neq 0$  gives  $y^* = 0$ ; however,  $z^* = 0$  gives  $y^* \neq 0$ , which means that the equilibrium solutions are confined to a plane. We take this plane to be the  $x^*y^*$ -plane. Thus, with  $y^* \neq 0$ , we obtain  $r_1 = r_2 = 1$ , which shows that the third body can be at either of these two points located at the same distance from the primaries as the two primaries are from each other [Roy, 2005]. Clearly, an equilateral triangle is formed and the two equilibrium points are called the Lagrange triangular  $L_4$  and  $L_5$  points (see Fig. 6). Both points are stable for the mass ratios  $0 \leq \mu \leq \mu_0$ , where  $\mu_0 = (1 - \sqrt{69}/9)/2$  is called Routh's critical mass ratio [Bardin, 2002]. Now, with  $y^* = z^* = 0$ , it is easy to show that Eq. (39) has three equilibrium solutions located on the line passing through the primaries. These three solutions are called the Lagrange collinear  $L_1$ ,  $L_2$  and  $L_3$  points (see Fig. 6), and they are unstable for any value of  $\mu$  [Ragos et al., 2000]. The Lagrange points are also called the libration points.

### 6.3. Jacobi's integral and constant

Let us introduce a potential function  $\phi^*$  defined as

$$\phi^* = \frac{1}{2} \left[ (x^*)^2 + (y^*)^2 \right] + \frac{\alpha}{r_1} + \frac{\mu}{r_2} , \qquad (42)$$

and write Eqs. (36), (37) and (38) in terms of  $\phi^*$ ,

$$\ddot{x}^* - 2\dot{y}^* = \frac{\partial \phi^*}{\partial x^*} \,, \tag{43}$$

$$\ddot{y}^* + 2\dot{x}^* = \frac{\partial \phi^*}{\partial y^*} \,, \tag{44}$$

and

$$\ddot{z}^* = \frac{\partial \phi^*}{\partial z^*} \ . \tag{45}$$

We multiply Eqs. (43), (44) and (45) by  $\dot{x}^*$ ,  $\dot{y}^*$  and  $\dot{z}^*$ , respectively, sum them up, and integrate them to obtain the following Jacobi integral

$$(v^*)^2 = 2\phi^* - C^* , (46)$$

where  $(v^*)^2 = (\dot{x}^*)^2 + (\dot{y}^*)^2 + (\dot{z}^*)^2$ , and  $C^*$  is the Jacobi constant in the rotating coordinate system given by

$$C^* = \mu + 2\mu\rho_0 + \frac{1-\mu}{\rho_0} + \frac{1\mu}{1+\rho_0} + 2\sqrt{\rho_0(1-\mu)} . \tag{47}$$

Here  $\rho_0 = R_0/D$  (see Fig. 5) represents the normalized initial position of the third mass.

Having obtained the Jacobi integral in the rotating coordinate system, we also write its explicit form in the non-rotating coordinate system

$$v^{2} - 2(x\dot{y} - y\dot{x}) = 2\left(\frac{\alpha}{r_{1}} + \frac{\mu}{r_{2}}\right) - C, \qquad (48)$$

where  $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$ , and C is the Jacobi constant in the non-rotating coordinate system.

# 6.4. Zero velocity curves

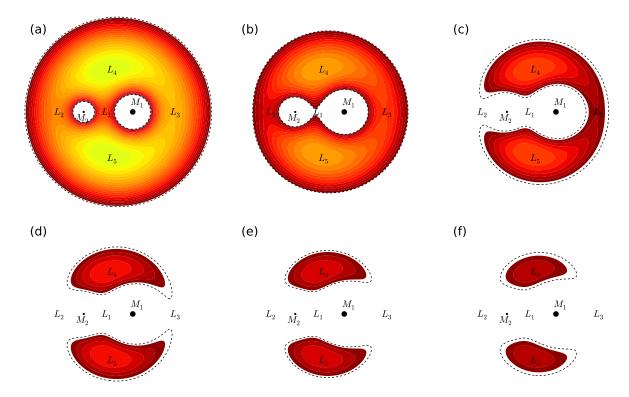
In the case when the particle's velocity  $v^*$  is zero, the Jacobi integral given by Eq. (46) reduces to  $2\phi^* = C^*$ , which defines the zero velocity curves (ZVCs) for the third body. Once  $C^*$  is determined by the initial conditions, the ZVC encloses a region in space where  $2\phi^* > C^*$  and where the third body must be found. The third body cannot be outside of this region because its velocity would be imaginary since  $2\phi^* < C^*$ . The ZVCs do not give any information about orbits of the third body in the region of space where it is allowed to move.

The shape of the ZVCs is uniquely determined by the values of  $C^*$ . Changes in the shape of the ZVCs in the  $x^*y^*$  plane with changing  $C^*$  expressed in terms of the initial position of the third mass  $\rho_0$  (see Eq. 47) are plotted in Fig. 6; the region inaccessible to the third mass is colored. The presented results were obtained for  $\mu = 0.3$  and  $\rho_0$  ranging from 0.2, which corresponds to a large value of  $C^*$ , to 0.6, which corresponds to a small value of  $C^*$ . Note that the regions inaccessible to the third mass shrink and eventually vanish at the Lagrange points  $L_4$  and  $L_5$ , as shown in Figs. 6d,e,f.

Transit orbits in the CR3BP are the trajectories that pass through the neck region of the ZVCs, which occurs in the vicinity of the Lagrange points  $L_1$ ,  $L_2$ , and  $L_3$  where the ZVCs open at these libration points at certain critical values of  $C^*$  (see Fig. 6). Moving along these transit orbits, the third body can transit between the primaries or even escape from the system. Extensive numerical studies of transit orbits in the CR3BP were recently performed by Ren and Shan [2012]. They established the necessary and sufficient conditions for such transition by using manifolds of the vertical and horizontal Lyapunov orbits along with the transit cones, and applied the results to the Sun-Earth CR3BP.

#### 6.5. Periodic and quasi-periodic orbits

Poincaré [1892] attached great importance to periodic orbits in the CR3BP, and he used his method (see Section 3.2) to identify and classify periodic motions of the third body



**Figure 6.** Zero velocity curves in the synodic coordinate system, computed for a mass ratio  $\mu=0.3$  and various initial positions  $\rho_o=R_o/D$  of the third mass. The dashed lines represent the boundary of zero velocity while the coloured region indicates the forbidden zones. (a)  $\rho_o=0.2$ ; (b)  $\rho_o=0.28$ ; (c)  $\rho_o=0.36$ ; (d)  $\rho_o=0.44$ ; (e)  $\rho_o=0.52$ ; (f)  $\rho_o=0.6$ ;

with respect to the synodic coordinate system. He searched for periodic orbits by taking  $\mu=0$  and then used analytical continuation to find periodic orbits for  $\mu>0$ . In his classification of periodic orbits, premiére sorte contains the orbits generated from the known two-body circular orbits; deuxiéme sorte contains the orbits generated from the known two-body elliptical orbits; and troisiéme sorte contains the orbits generated from the known two-body circular orbits, but with a non-zero inclination of the third body with respect to the plane in which the orbits of the primaries are confined.

Poincaré's work on periodic orbit was continued by Darwin [1897, 1909], Moulton et al. [1920] and Strömgren and Pedersen [1922]. Then, Strömgren [1935] performed extensive studies of the periodic orbits for  $\mu = 0.5$  and demonstrated that they belong to the *premiére sorte*. He concluded that the termination of this family of orbits must be an asymptotic periodic orbit spiraling into the Lagrange points  $L_4$  and  $L_5$ ; a proof of this conjecture was given by Henrard [1973] and Buffoni [1999].

Various families of periodic orbits in the commensurable 3D CR3BP, where two bodies are close to commensurability in their mean motions around the central body, were identified and classified by Sinclair [1970]. Moreover, a special class of periodic orbits in the CR3BP resulting from an analytical continuation of Keplerian rectilinear

periodic motions was discovered by Kurcheeva [1973], who showed that the period of such orbits is an analytical function of  $\mu$ ; the conditions when these orbits are stable were established analytically by Ahmad [1995]. Comprehensive studies of the periodic orbits in the CR3BP were performed by Hénon [1965, 1974] who classified them and also investigated intersections between families of generating orbits [Hénon, 1997, 2001].

There are periodic orbits around the Lagrange points as demonstrated by Strömgren [1935]. Periodic orbits in the vicinity of these points are typically classified as halo, vertical, and horizontal Lyapunov orbits [Henrard and Navarro, 2004]. They arise either as a continuation of infinitesimal oscillations or as bifurcations of the planar periodic orbits around the points. Some unstable periodic orbits around  $L_1$ ,  $L_2$  and  $L_3$  were identified and classified by Zagouras and Kazantzis [1979] for  $\mu = 0.00095$  (Sun-Jupiter). Since  $L_4$  and  $L_5$  are stable [Szebehely, 1967, Roy, 2005], there are stable zones around these points for certain physical conditions [Dvorak and Lohinger, 1991]. The existence of periodic orbits in the vicinity of the  $L_4$  and  $L_5$  have been studied. Perdios et al. [1991] identified the long and short periodic orbits near the two libration points for the mass parameter  $\mu$  ranging from 0.03 to 0.5, and Markellos [1991] found families of remarkable termination orbits. According to Routh's criterion, the existence of periodic orbits near the triangular libration points is limited to the mass ratios  $0 \le \mu < \mu_0$ , where  $\mu_0$  is Routh's critical mass ratio [Bardin, 2002]. Moreover, Broucke [1999] adopted canonical units to define the unit circle and formulated a symmetry theorem that was used to discover new long and short-period orbits around  $L_4$  and  $L_5$ .

All orbits discovered in the above studies were symmetric periodic orbits, which are much more easy to generate than non-symmetric periodic orbits. Systematic explorations of non-symmetric periodic orbits in the 3D CR3BP were done by Zagouras and Markellos [1977] and Zagouras et al. [1996], who discovered new orbits and classified them. Searches for non-symmetric periodic orbits near  $L_4$  (because of the symmetry conditions the orbits are the same at  $L_5$ ) were performed by Henrard [1970, 2002], who discovered an infinite number of families of non-symmetric periodic orbits in the vicinity of the points. In recent work of Henrard and Navarro [2004] showed that some of these orbits are emanating from homoclinic orbits (see Section 6.7).

The existence of quasi-periodic orbits around the libration points was confirmed by using different semi-analytical, numerical, and combined analytical-numerical methods [Ragos et al., 1997, Jorba and Masdemont, 1999, Gómez et al., 1999]. An overview of quasi-periodic orbits around  $L_2$  is given by Henrard and Navarro [2004], who showed that different families of such orbits exist and that these families are associated with the known halo, vertical and horizontal Lyapunov orbits. They also presented a fast numerical method based on multiple Poincaré sections (see Section 6.7) and used it to find quasi-halo periodic orbits. Henrard and Navarro [2004] demonstrated that the method gives full convergence for a given family of quasi-periodic orbits, and that it is robust near chaotic regions.

## 6.6. Circular Hill problem

In the Hill three-body problem, the mass of one primary dominates over the other masses  $(M_1 >> M_2)$ , and the mass of the third body is negligible when compared to both primaries  $(M_1 >> M_3)$  and  $M_2 >> M_3$ . Because of these mass relationships,  $M_2$  moves around  $M_1$ , and if its orbit is circular, the problem is called the *circular* Hill problem. The third body can move either in 3D space or in the same plane as the circular orbit, and its orbit is determined by the equations of motion governing the problem [Szebehely, 1967]. The equations of motion in the non-rotating coordinate system are obtained by taking  $\mu \to 0$  in Eqs (33) through (35), and in the synodic coordinate system by applying the same limit to Eqs (36) through (38).

An extensive discussion of the *circular* Hill problem and its specific applications to the Sun-Earth-Moon system is given by Szebehely [1967], who also presents families of periodic orbits, the libration points and the ZVCs. Applications of the *circular* Hill problem to the Sun-Jupiter-asteroid system was considered by Bolotin and MacKay [2000], who proved the existence of an infinite number of periodic and chaotic (almost collision) orbits in the synodic coordinate system. Moreover, some periodic orbits were identified by Zagouras and Markellos [1985], and Hénon [2003] extended his previous work [Hénon, 1974] on the *circular* Hill problem and discovered new families of double and triple-periodic symmetric orbits.

# 6.7. Poincaré's qualitative methods and chaos

Poincaré [1892] considered an unstable periodic solution to the CR3BP (or in some specific cases to the circular Hill problem) and represented it by a closed curve. He also accounted for a family of asymptotic solutions located on two asymptotic surfaces. The closed curve and the two asymptotic surfaces were his geometrical representations of the CR3BP [Barrow-Green, 1997]. Poincaré demonstrated that the equations describing these surfaces could be written as

$$\frac{x_2}{x_1} = f(y_1, y_2) , (49)$$

where the asymptotic series for the first and second surface are given by  $x_1 = s_1(y_1, y_2, \sqrt{\mu})$  and  $x_2 = s_2(y_1, y_2, \sqrt{\mu})$ , respectively. In addition, the series must satisfy

$$\frac{\partial F}{\partial x_1} \frac{\partial x_i}{\partial y_1} + \frac{\partial F}{\partial x_2} \frac{\partial x_i}{\partial y_2} + \frac{\partial F}{\partial y_i} = 0$$
 (50)

where i = 1 and 2, and  $F = F_0 + \mu F_1 + \mu F_2 + ...$ , with  $F_0 \neq F_0(y)$  but  $F_1, F_2, ...$  being functions of both x and y; see the text below Eq. (12).

Poincaré assumed that there were certain values  $x_i^0$  for which  $\partial F/\partial x_i$  are commensurable, which allows writing  $x_i = \Phi_i(y_1, y_2)$  with  $F(\Phi_1, \Phi_2, y_1, y_2) = \tilde{C}$  satisfying Eq. (50). To integrate this equation, Poincaré considered  $x_i = x_i^0 + (\mu x_i^1)^{1/2} + \mu x_i^2 + ...$ , and determined the coefficients  $x_i^k$ , with k = 1, 2, ... The purpose of the second

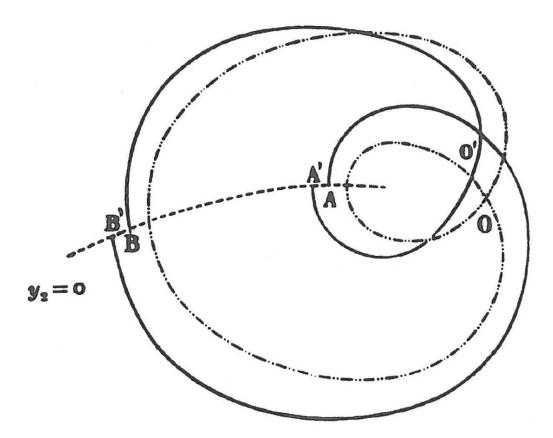


Figure 7. Intersection of the asymptotic interfaces with the transverse section  $y_1 = 0$  as explained in the main text. Reproduced after Barrow-Green [1994]. Copyright 1994 Springer.

approximation was to evaluate an arbitrary number of coefficients of the above series, and to write the approximate equations for the asymptotic surfaces. Finally, in the third approximation he constructed the intersection of the asymptotic interfaces with the transverse section  $y_1 = 0$ , and returned to his geometrical description, as depicted in Fig. 7.

In Fig. 7, the closed curve represented by the dotted and dashed line corresponds to the generating (unstable) periodic orbit. In addition, the curves AO'B' and A'O'B represent the asymptotic surfaces and their intersection with  $y_1 = 0$ , and the dashed line represents the curve  $y_1 = y_2 = 0$ . Poincaré realized that the curves AO'B' and A'O'B could not be closed. He considered possible trajectories that would simultaneously belong to both sides of the asymptotic surface and called them *doubly asymptotic* or homoclinic orbits [Anderson, 1994].

Poincaré identified primary, secondary, tertiary, and quaternary homoclinic orbits and realized that there could be an infinite number of them leading to a homoclinic tangle, which appeared to be the first mathematical manifestation of chaos in the CR3BP; knowing how complicated the tangle could be, he did not attempt to draw it. Chaos has a sensitive dependence on initial conditions, and its existence in the CR3BP was confirmed by computational simulations of the CR3BP performed

by Smith and Szebehely [1993], Koleman et al. [2012], and many others. Moreover, Vela-Arevalo and Mardsen [2004] demonstrated that a method based on time-varying frequencies is a good diagnostic tool to distinguish chaotic trajectories from regular ones.

Some of the modern computational methods use Poincaré sections to identify quasiperiodic and chaotic orbits. Poincaré [1892] studied trajectories close to a periodic orbit
and investigated their return to a line drawn across the orbit (now known as Poincaré
section). The behavior of trajectories returning to the line forms the first return map,
or Poincaré's map, and he used it to reduce the dimension of phase space of the threebody problem by one. He also showed that solutions could leave two equilibria and then
return to them asymptotically, thus forming a heteroclinic orbit. The above concepts
originally introduced by Poincaré are now commonly used in modern theories of chaos
[Thompson and Stewart, 1986, Hilborn, 1994, Musielak and Musielak, 2009].

# 7. The elliptic restricted three-body problem

## 7.1. Basic equations and integrals of motion

Similar to the CR3BP, the elliptic restricted three-body problem (ER3BP) requires that the two objects (called the primaries) have their masses significantly larger than the third one  $(M_1 >> M_3 \text{ and } M_2 >> M_3)$ , and that the third mass has no gravitational influence on the primaries. However, the main difference between the CR3BP and ER3BP is that, in the latter, motions of the primaries are along elliptical orbits around their center of mass, which has important implications for the mathematical description of the system as there is not a Jacobi's constant in the ER3BP [Szebehely, 1967, Marchal, 1990, Palacián et al., 2006].

In general, the third mass can move in 3D space, a case is called the 3D ER3BP. However, if motions of the third body are restricted to the same plane as the primaries, we have the planar ER3BP. Typically, the governing equations for the 3D ER3BP are presented in the rotating-pulsating coordinates, which allowed for an elegant simplification of the problem [Szebehely, 1967, Marchal, 1990]. In the following, we describe the governing equations of the 3D ER3BP, but then we make clear distinctions between the 3D and planar ER3BP when discussing some recently obtained results.

Let  $M_2$  move around  $M_1$  on an elliptic orbit with eccentricity e, true anomaly f and semimajor axis a = 1, and let  $\chi, \eta, \xi$  be the rotating-pulsating coordinates. With f being an independent variable, the set of governing equations describing the 3D ER3BP can be written [Szebehely, 1967] in the following form:

$$\frac{\partial^2 \chi}{\partial f^2} - 2 \frac{\partial \eta}{\partial f} = \omega_{\chi} , \qquad (51)$$

$$\frac{\partial^2 \eta}{\partial f^2} + 2 \frac{\partial \chi}{\partial f} = \omega_{\eta} , \qquad (52)$$

and

$$\frac{\partial^2 \xi}{\partial f^2} + \frac{\partial \xi}{\partial f} = \omega_{\xi} , \qquad (53)$$

where  $\omega = \Omega/(1 + e \cos f)$  with  $\Omega = (\chi^2 + \eta^2 + \xi^2)/2 + (1 - \mu)/r_1 + \mu/r_2 + \mu(1 - \mu)/2$ ,  $r_1^2 = (\chi - \mu)^2 + \eta^2 + \xi^2$  and  $r_1^2 = (\chi - \mu + 1)^2 + \eta^2 + \xi^2$ . For the planar ER3BP, it is necessary to take  $\xi = 0$ .

We now follow Llibre and Pinol [1990] and introduce  $q_1 = -\chi + \mu$ ,  $q_2 = -\eta$ ,  $q_3 = \xi$ ,  $p_1 = -\chi' + \eta$ ,  $p_2 = -\eta' + \mu$  and  $p_3 = \xi'$ , where prime indicates d/df. Then Eqs (51) through (53) can be written as

$$\frac{dq_i}{df} = \frac{\partial H}{\partial p_i} , \qquad \frac{dp_i}{df} = -\frac{\partial H}{\partial q_i} , \qquad (54)$$

where i = 1, 2 and 3, and H is the time dependent Hamiltonian  $H = H_0 + \mu H_1$  with

$$H_0 = \frac{1}{2} [(p_1 + q_2)^2 + (p_2 - q_1)^2 + p_3^2 + q_3^2] - \frac{1}{1 + e \cos f} \left[ \frac{1}{2} \left( q_1^2 + q_2^2 + q_3^2 \right) + \frac{1}{r_1} \right] , \quad (55)$$

and

$$H_1 = \frac{1}{1 + e\cos f} \left( q_1^2 + \frac{1}{r_1} - \frac{1}{r_2} - \frac{1}{2} \right) . \tag{56}$$

Since H is an explicit function of f, it is not an integral of motion for the ER3BP; however, H is an integral of motion for the CR3BP for which e = 0. The planar ER3BP requires that  $q_3 = p_3 = 0$ .

We may now apply the procedure of deriving the Jacobi integral for the CR3BP (see Section 6.3) to Eqs (51) through (53), which are the ER3BP governing equations. The procedure requires multiplying Eqs (51), (52), and (53) by  $\chi'$ ,  $\eta'$ , and  $\xi'$ , respectively, adding them and integrating. The result is

$$\left(\frac{\partial \chi}{\partial f}\right)^2 + \left(\frac{\partial \eta}{\partial f}\right)^2 + \left(\frac{\partial \xi}{\partial f}\right)^2 = 2\int (\omega_\chi d\chi + \omega_\eta d\eta + \omega_\xi d\xi) ,$$
(57)

which is equivalent to the Jacobi integral in the CR3BP; however, the difference is that the RHS of the above equation is not constant. Thus, the Jacobi integral for the ER3BP is not a constant of motion.

There are other constants of motions for the ER3BP. As originally pointed out by Contopoulos [1967], two integrals of motion for the planar ER3BP exist if the system is considered in rotating coordinates with their x-axis passing through the primaries. The results were generalized to the 3D ER3BP by Sarris [1982], who found three integrals of motion for a small eccentricity of the relative orbit of the primaries, and for a small distance of the third body from one of the primaries, and demonstrated that they depended periodically on time.

# 7.2. Periodic solutions, libration points and Lie series solutions

Periodic orbits showing chains of collision orbits of the original Kepler problem were first studied by Poincaré [1892], who named them the *second species solutions*. Searches for such orbits in the 3D ER3BP were performed by Gomez and Olle [1991a], Gomez and Olle [1991b], Bertotti [1991], Bolotin and MacKay [2000], Bolotin [2005] and

Palacián and Yanguas [2006]. In Bertotti's work, a variational method designed to find such periodic orbits was developed. Bolotin [2005] considered the planar ER3BP with a small mass ratio and eccentricity. Then he proved the existence of many second species periodic orbits. The fact that there are ejection-collision orbits in the 3D ER3BP was demonstrated by Llibre and Pinol [1990]. Moreover, the effectiveness of the phenomenon of the gravitational capture of the third body by the primaries in the 3D ER3BP with small  $\mu$  was discussed by Makó and Szenkovits [2004].

Sinclair [1970] identified and classified various families of periodic orbits in the commensurable planar ER3BP, where two bodies are close to commensurability in their mean motions taking place in the same plane around the central body. Moreover, Hadjidemetriou [1992] and Haghighipour et al. [2003] discovered new families of stable 3:1 and 1:2 periodic orbits in the planar ER3BP.

In addition to the existence of periodic orbits in the 3D and planar ER3BP, there are also the equilibrium point solutions or the libration points in both problems. The conditions that define the libration points are obtained from Eqs (51) through (53), namely,  $\chi'' = \eta'' = \xi'' = \chi' = \eta' = \xi' = 0$  and  $\partial\Omega/\partial\chi = \partial\Omega/\partial\eta = \partial\Omega/\partial\xi = 0$ . Using these conditions, it is easy to show that there are five libration points in the ER3BP (similar to the CR3BP), and that three of them are collinear and two are triangular or equilateral [Danby, 1964, Bennett, 1965]. Motions around these libration points were investigated by Szebehely [1967], who demonstrated that the triangular points pulsate together with their own coordinate systems, and by Choudhry [1977], who showed that the coordinates of the libration points depend explicitly on time.

A solution to the planar ER3BP was constructed by Delva [1984], who used the method of Lie series that is another form of the Taylor series [Schneider, 1979]. The ER3BP in the rotating-pulsating coordinate system was considered and the Lie operator for the motion of the third body was derived as a function of coordinates, velocities, and true anomaly of the primaries [Delva, 1984]. An analytical form of the Lie series was obtained and used to compute orbits; an extension of this method to the 3D ER3BP should be straightforward. Numerical solutions to both the 3D and planar ER3BP were obtained and some specific examples are presented in the next section devoted to applications.

#### 7.3. Elliptic Hill problem

We now briefly discuss the elliptic Hill problem and its potential applications to astrodynamics and extrasolar planetary systems. As stated in Sections 1 and 6.7, in the three-body Hill problem  $M_1 >> M_2 >> M_3$ . If the orbit of  $M_2$  around  $M_1$  is elliptic, we then have the elliptic Hill problem [Ichtiaroglou, 1980]. A few families of periodic orbits in the elliptic Hill problem were discovered by Ichtiaroglou [1981], who found all of them to be highly unstable. In more recent work, Voyatzis et al. [2012] calculated a large set of families of periodic orbits in the *elliptic* Hill problem, determined their stability and applied the results to motions of a satellite around a planet. Szenkovits and Makó

[2008] established a more accurate criterion of the Hill stability and used it to investigate stability of exoplanets in the systems:  $\gamma$  Cephei Ab, Gliese 86 Ab, HD 41004 Ab, and HD 41004 Bb. They concluded that the orbits of these exoplanets will remain stable, according to the Hill stability criterion, in the next few million years.

# 8. Applications of the restricted three-body problems

Among the possible solutions of the three-body problem, the simplification imposed in the restricted three-body problem has been the most practical, and therefore many different astronomical systems have been studied within this framework. We present applications of the restricted three-body problem with an emphasis on extrasolar planets and moons. Since orbital dynamics plays a role in determining the likelihood of a planet being classified as habitable, residing in a region of space in which liquid water could persist on the surface of the planet, we provide additional discussion within that context.

## 8.1. The Earth-Moon system with a spacecraft

The advent of the space exploration program presented a host of challenging three-body problems, and it established a new area of study in astrodynamics. The initial problem was that of solving the motion of artificial satellites, followed by the task of determining spacecraft orbital maneuvers required for missions to the Moon.

The orbital dynamics of artificial satellites and spacecraft is based on n-body theory, but it can be approximated as a three-body problem. Moreover, we distinguish between spacecraft trajectories and motion of artificial satellites (orbiters, space telescopes, space stations, communications satellites), as the solution for each case have different requirements and outcomes. The following are some interesting examples:

- (i) For analysis of space missions, such a spacecraft moving between two planets, or from Earth to the Moon, we may apply the restricted three-body problem. However, space missions through the farther regions of the Solar System are rather complex and require that spacecraft trajectories be designed by decoupling an n-body system into several three-body systems. For example, by decoupling the Jovian moon n-body system into several three-body systems, it is possible to design a spacecraft orbit which follows a prescribed itinerary in its visit to Jupiter and its moons. In fact, the Jupiter-Ganymede-Europa-spacecraft system is approximated as two coupled planar three-body systems [Gómez et al., 2001]. Furthermore, for a spacecraft in transit from Earth to the outer planets, its flight may involve a gravity assist, which yields a four-body problem; once far away from the Earth, it reduces to a three-body problem. Further complications arise when spacecraft move in the vicinity of asteroids or other celestial objects with gravitational fields that are not sufficiently well-established.
- (ii) For artificial Earth satellites, the problem of the orbit evolution is an instance of the four-body problem. The evolution of a satellites orbit with a high eccentricity, such as that of the Milniya communication satellites, is very different from that of a

low-Earth orbit of low eccentricity (e.g., Hubble Space Telescope). In the latter case the most important perturbations are due to the oblateness of the Earth and the air drag. Both effects are small for a moderate to high eccentricity orbit with a large semi-major axis. But the gravitational influence of the Sun and the Moon becomes predominant. The Kepler Space Telescope orbits the Sun, which avoids the gravitational perturbations and torques inherent in an Earth orbit.

In general, the design of space missions requires analytical and numerical methods to solve planar circular restricted three-body problem (PCR3BP) and circular restricted three-body problem (CR3BP). For example, when decomposing an n-body system into three-body systems that are not co-planar, such as the Earth-Sun-spacecraft and Earth-Moon-spacecraft systems, we need to consider three-dimensional configurations and solve circular restricted three-body problems (CR3BP). The planar motion of a spacecraft relative to two other masses which orbit each other is very complex, with nonlinear effects due to the perturbation of the two bodies, for which no general analytical solution exists. Therefore, the orbital analysis for such a spacecraft is based on sophisticated numerical methods.

## 8.2. The Sun-Earth-Moon system

This particular solution of the three-body problem has influenced the theory of gravity since its inception by Newton [1687]. In comparison to the previous application concerning satellites, this approach relies on the Sun-Earth system as the two dominant bodies with the Moon as a test particle. This was effectively the approach taken by Newton; however, he was not successful in determining stable solutions ( $\sim 8\%$  uncertainties). The problem is that the mass of the Moon is comparable to that of the Earth within two orders of magnitude; it is also relatively close to the Earth, therefore, inducing reaction forces and tides that are not accounted for in the restricted three-body problem.

Others have made progress in determining stability conditions for the Sun-Earth-Moon system. Hill [1877, 1878] developed a general stability criterion based upon the balance of gravitational forces on a test particle between the Earth and the Sun. This criterion has been refined statistically through numerical simulation with estimates indicating a limit of  $\sim 3.2R_H$ , where  $R_H$  denotes the Hill Radius given by  $R_H = (m_{\oplus}/3M_{\odot})^{1/3} a$  [Gladman, 1993]. Limitations of this criterion in applications two extrasolar planetary systems were pointed out by Cuntz and Yeager [2009]. Recently, Satyal et al. [2013] used a method proposed by Szenkovits and Makó [2008] that utilizes the Hill stability criterion; we discuss this approach in Section 8.6.

#### 8.3. The Sun-Jupiter system with an asteroid

Beyond the Sun-Earth-Moon system, the next most studied restricted three-body problem is the Sun-Jupiter-asteroid system. Since masses of asteroids are very small when compared to either the Sun or Jupiter, their description fits well within the framework of the restricted three-body problem. Specific applications include dynamics of asteroids within the Asteroid Belt, and the origin of the Kirkwood Gaps related to a depletion of the asteroid population in some orbits due to resonances with Jupiter. Resonances of asteroids with Jupiter have been studied by Nesvorný et al. [2002], including possible three-body resonances in the Solar System.

As discussed in Section 6.2, the collinear Lagrange points are unstable. Thus, no population of asteroids in the vicinity of these points is expected. However, there is a population of asteroids with stable orbits around the  $L_4$  and  $L_5$  Lagrange points. These populations are the Trojan asteroids, and their stability has been investigated using the restricted three-body formulation [Beutler, 2005, Valtonen and Karttunen, 2006].

## 8.4. A single star with giant and terrestrial exoplanets

Since the confirmation of the first exoplanet 51 Pegasi around a solar-type star [Mayor and Queloz, 1995, Marcy and Butler, 1995], we now know almost 1800 exoplanets, including more than 1000 exoplanets discovered by the NASA's *Kepler* space telescope; this number of discovered planets is growing everyday ¶. Prior to the launch of *Kepler*, most known exoplanets were Jupiter-analogs with respect to their sizes and masses. A very small subset of these systems are now known to host terrestrial planets. Thus, the restricted three-body problem has been used to investigate a planetary system with a single star, gas giant, and terrestrial exoplanets.

In studies performed by Noble et al. [2002], broad regions of orbital stability were found for terrestrial exoplanets located in the habitable zone (HZ) of 51 Pegasi and other planetary systems. More general stability limits for extra-solar planetary systems with two exoplanets were established by Barnes and Greenberg [2006]. The HZ's for main-sequence stars of different spectral stars were originally established by Kasting et al. [1993]. Prospects for the existence of a terrestrial planet in stellar HZ with known giant exoplanets were discussed by Jones et al. [2005]. In more recent work, Kopparapu and Barnes [2010] identified 4 extra-solar planetary systems that can support a terrestrial planet in their HZ's.

In the recent work of Goldreich and Schlichting [2014], resonances in the planar, circular, restricted three-body problem with possibilities of semimajor axis migration and eccentricity damping result from planet-disk interactions. The authors showed that the migration and damping results in an equilibrium eccentricity, and that librations around this equilibrium are overstable. Applications of these results to exoplanets discovered by the *Kepler* satellite demonstrated that most of the observed planets are overstable, which leads to passage through resonance. A more realistic case of the planar, three-body problem is also considered with two (inner and outer) exoplanets being in resonance, and the departure from exact resonance is discussed.

The most prolific detection method before *Kepler* was the radial velocity method, which relies on shifts of lines within the spectra of the host star. These shifts demonstrate

the reflex motion of the star due to the gravitational force exerted by the unseen companion. Terrestrial exoplanets would induce shifts in the radial velocity of the host star on the order of centimeters per second, which is just beyond today's limits for this detection technique. In terms of exoplanet detections, the photometric transit method has uncovered through the *Kepler* sample, that giant and terrestrial planets can exist within the same system with a larger fraction of planets being Neptune-sized or smaller [Howard et al., 2012, Fressin et al., 2013, Santerne et al., 2013]. In such systems the restricted three-body problem can be used to constrain the likelihood of additional planets.

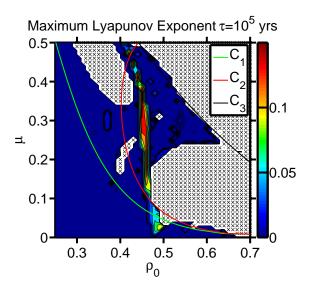
## 8.5. A single star with a giant exoplanet and exomoon

Another set of systems has been proposed for the Jupiter-like planets discovered within a host star's HZ [Kasting et al., 1993, Kopparapu and Barnes, 2010]. Giant exoplanets within this region present a major challenge to the search for terrestrial exoplanets, which could support life. It has been shown that giant exoplanets near the HZ can induce perturbations upon the putative terrestrial exoplanet, thereby making the orbit of the smaller body unstable. In order to keep the search for habitability meaningful, exomoons have been considered as possible abodes for life, which orbit giant exoplanets within the HZ. Studies of such systems are based on the restricted three-body problem.

The plausibility of a Jupiter-like exoplanet located in the habitable zone to host an Earth-mass moon has been considered [Williams et al., 1997]. The formation of such a system may experience considerable difficulties [Canup and Ward, 2006], but the capture of a terrestrial planet still remains possible [Kipping, 2009, Kipping et al., 2012, 2013b,a]. Other studies have used a parameter survey approach to determine the stability of possible exomoons; see applications to the system HD 23079 system [Cuntz et al., 2013]. Studies of such systems are based on the formulation of the restricted three-body problem.

#### 8.6. Binary stellar systems with a giant or terrestrial exoplanet

One of the most extreme cases of the restricted three-body problem involves the study of exoplanets orbiting either one or both components of a binary star system, in which the more massive and less massive stars are called the primary and secondary, respectively. The categorization of these systems has been referred to as satellite-like (S-type) or planetary-like (P-type), where the former contains a planet that orbits a single star and the latter orbits both stars [Dvorak, 1982]; the extent of HZ's in such systems have been recently established [Eggl et al., 2012, Kane and Hinkel, 2013, Kaltenegger and Haghighipour, 2013, Haghighipour and Kaltenegger, 2013, Cuntz, 2014]. Using this convention, many different systems have been studied based on the restricted three-body problem, with various methods including topology and chaos indicators. Furthermore, planetary formation in these systems have been studied even before one P-type system (Kepler16) with the exoplanet had been rigorously



**Figure 8.** Determination of stability of S-type planets using the method of Lyapunov exponents. Regions of stability (blue) are indicated as well as regions of chaos (red). Areas with a white background denote unstable regions on 0.1 Myr timescales. Reproduced after Quarles et al. [2011]. Copyright 2011 Astronomy & Astrophysics.

confirmed by Doyle et al. [2011]. As this field encompasses a separate research area [Pilat-Lohinger and Funk, 2010], we now discuss only selected recent approaches.

Using a topological method, Musielak et al. [2005] and Cuntz et al. [2007] determined stability criteria for the special case of binaries hosting exoplanets, which relied upon the Jacobi constant formalism in contrast to previous statistical approaches [Holman and Wiegert, 1999, Dvorak, 1984]. This approach allowed for the exploration of marginally stable systems, which would have prematurely been characterized as stable using a parameter space consisting of the mass ratio  $\mu = M_2/(M_1 + M_2)$  and the initial distance ratio  $\rho_o = R/D$ , where  $M_2 \leq M_1$ , R is the distance from the planet to  $M_1$ , and D is the initial separation of the stellar components.

Eberle and Cuntz [2010a] furthered this exploration of determining stability of planets in stellar binary systems using Hamilton's Hodograph [Hamilton, 1847]. This method demonstrated how the effective eccentricity of an exoplanet would change due to interactions with both stellar components. Eberle and Cuntz [2010b] explored the same parameter space and produced a map of general stability with contours of the effective eccentricity. Quarles et al. [2011] used the parameter space along with the maximum Lyapunov exponent [Lyapunov, 1907], and found similar contours of stability with structures relating to orbital resonances and islands of instability.

Figure 8 shows the parameter space with stability contours within a linear regime. Quarles et al. [2011] showed that a 3:1 resonance can cause chaos within the circular restricted three body problem for a wide range of mass ratios. Although these results have been mainly theoretical, they have been applied to the limited number of exoplanets found in stellar binary systems. One such system,  $\nu$  Oct, has a controversial discovery as the proposed distance ratio of the exoplanet lies within a region of instability (see Fig.

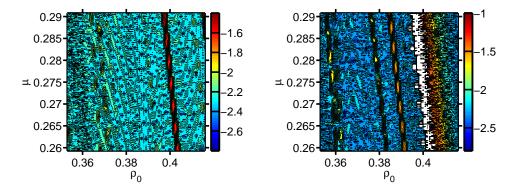
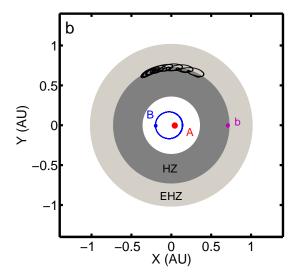


Figure 9. Determination of stability of the planet in  $\nu$  Octantis assuming a retrograde orbiting planet initially placed at apastron (left) and periastron (right). Reproduced after Quarles et al. [2012a]. Copyright 2012 Monthly Notices of the Royal Astronomical Society.

9). This posed a problem for the confirmation of the exoplanet. However, if one considers an exoplanet in retrograde orbit relative to the orbiting binary, then enlarged regions of stability arise, which would be consistent with the putative exoplanet [Eberle and Cuntz, 2010b, Quarles et al., 2012a, Goździewski et al., 2013].

The restricted three-body problem has also been applied to P-type or circumbinary exoplanets. Doyle et al. [2011] announced the first circumbinary exoplanet in the Kepler-16 system. This has been an important discovery for the theory behind the restricted three-body problem to be tested. Soon after the discovery was announced, Quarles et al. [2012b] investigated cases when an Earth-mass exoplanet could be injected into the system; the studies of orbital stability of such an exoplanet could help in discovering habitable Earth-analogs in the Kepler-16 system.

Since injected terrestrial planets in the Kepler-16 system constitutes a restricted four-body system, Quarles et al. [2012b] reduced it to a restricted three-body problem in order to determine the possibility of the existence of Trojan moons in this system. With the distance between the stellar hosts (denoted as A and B in Fig. 10), and the giant (Saturnian) exoplanet (denoted as b in Fig. 10) in the Kepler-16 system being relatively large, Earth-mass objects could remain in stable orbits around the approximated equilateral (L4 and L5) points in many different types of orbits. Note that the giant planet is located at the border between the HZ and the so-called extended habitable zone (EHZ), which requires more extreme exoplanetary atmosphere with a significant back-warming [Quarles et al., 2012b]. Figure 10 shows one of the orbits possible in this system; an exoplanet in this regime could be habitable, if it has a strong greenhouse effect [Cuntz, 2014]. However, either the formation or capture of such an exoplanet would be difficult to achieve within the environment of the circumbinary disk [Meschiari, 2012, Paardekooper et al., 2012].



**Figure 10.** Orbits of possible Trojan moons in Kepler-16 using a three-body approximation - see the main text for explanation. Reproduced after Quarles et al. [2012b]. Copyright 2012 The Astrophysical Journal.

## 9. The relativistic three-body problem

## 9.1. The Newton and Einstein theories of gravity

The three-body problem discussed in the previous sections is based on the Newtonian theory of gravity in which the bodies are assumed to be spherical and can be considered as point particles of given masses. Since Newton's description of gravity is only a weak field approximation of Einstein's General Theory of Relativity (GTR), which is currently our best theory of gravity, now we want to discuss implications of using GTR on the three-body problem. GTR replaces the flat space-time of Special Theory of Relativity by a 4D smooth, continuous, pseudo-Riemannian manifold endowed with the metric  $ds^2 = g_{\mu\nu}(x)dx^{\mu}dx^{\nu}$ , where  $g_{\mu\nu}$  is the metric tensor to be determined by Einstein's field equations,  $\mu$  and  $\nu$  are 0, 1, 2 and 3, and the usual summation conventions are employed [Hobson et al., 2006]. According to GTR, the 4D space-time is curved by the presence of bodies and the resulting curvature is identified with gravity, which clearly shows that Newton's and Einstein's theories of gravity are significantly different. Nevertheless, Einstein developed his GTR in such a way that in the limit of weak gravitational fields, Newton's description of gravity remains valid.

To account for relativistic effects on motions of astronomical bodies, relativistic celestial mechanics was formulated and its methods applied to many interesting astronomical problems, including the three-body problem [Brumberg, 1972, 1991, Kopeikin et al., 2011]. Since Einstein's field equations of GTR can only be solved for the simple one-body problem, and even the two-body problem cannot be solved rigorously in GTR, the equations of motion of the relativistic three bodies must be obtained with some approximations. Typically, it is the so-called post-Newtonian approximation, which is a weak field and slow motion approximation, that is used to derive the post-

Newtonian equations of motion of the three spherically symmetric bodies; the resulting equations depend not only on masses but also on physical parameters relevant to the internal structure of the bodies, such as elastic energy, moment of inertia, among others [Brumberg, 1991]. According to Kopeikin et al. [2011] and references therein, the presence of these parameters on the motions of the three bodies may actually be irrelevant. Thus, some authors prefer to use the Einstein-Infeld-Hoffmann (EIH) equation of motion, which is valid for point-like masses with no internal structure [Einstein et al., 1938, Infel, 1957].

## 9.2. The general relativistic three-body problem

In the general relativistic three-body problem, there is no restriction on masses, except that they are finite. The post-Newtonian equations of motion for this problem were obtained by Brumberg [1972] and applied to different astronomical settings by Brumberg [1991] as well as to the Solar System by Kopeikin et al. [2011]. Hamiltonian and standard formalisms have also been formulated for the consideration of general relativistic effects within an n-body framework [Saha and Tremaine, 1992, 1994, Kidder, 1995].

The existence of the post-Newtonian collinear solution, which is a relativistic extension of the original Euler collinear solution (see Section 3.1 and Fig. 1), was shown by Yamada and Asada [2010a] who used the EIH equation of motion. They also calculated the relativistic corrections to the spatial separation between the three bodies required for the solution to be valid. The uniqueness of the relativistic collinear solution was demonstrated by Yamada and Asada [2010b].

Moreover, a triangular solution that is a relativistic version of the Lagrange solution with an equilateral triangle joining the three bodies (see Section 3.1 and Fig. 2) was originally obtained by Krefetz [1967]. More recently, Ichita et al. [2011] studied the post-Newtonian effects on this solution and showed that the solution satisfies the post-Newtonian equations of motion only when all three masses are equal. Thereafter, it was demonstrated by Yamada and Asada [2012] that the relativistic corrections to each side of the triangle must be added and, as a result, the relativistic triangle will not always be equilateral.

Another interesting result is that the 8-type periodic solution for the general (non-relativistic) three-body problem with equal masses (see Section 3.2 and Fig. 3a) does also exist in the general relativistic three-body problem. Its first and second post-Newtonian orders were investigated numerically by Imai et al. [2007] and Lousto and Nakano [2008], respectively, who demonstrated that the relativistic effects are responsible for only small changes in the shape of this 8-type solution.

It is also likely that relativistic versions of the 13 new periodic orbits found by Šuvakov and Dmitrašinović [2013] in the *planar general* (non-relativistic) three-body problem (see Section 3.2 and Fig. 3) will be discovered and that relativistic corrections to the results presented in Fig. 3 will be found. However, at the present time this idea

still remains to be a conjecture.

# 9.3. The restricted relativistic three-body problem

In the restricted relativistic three-body problem (RR3BP), the two primaries have dominant masses and move around their center of mass; however, the third mass is very small and its gravitational influence on the primaries is negligible. The post-Newtonian equations describing motions of the third body in the RR3BP were originally obtained by Brumberg [1972], who used a synodic (rotating) frame of reference with the origin at the center of mass and the primaries fixed.

Let us consider a planar RR3BP, introduce the synodic coordinates v and  $\zeta$ , and follow Douskos and Perdios [2002] and write the post-Newtonian equations of motion in the following form

$$\ddot{v} - 2n\dot{\zeta} = \frac{\partial U}{\partial v} - \frac{d}{dt} \left( \frac{\partial U}{\partial \dot{v}} \right) , \qquad (58)$$

and

$$\ddot{\zeta} + 2n\dot{v} = \frac{\partial U}{\partial \zeta} - \frac{d}{dt} \left( \frac{\partial U}{\partial \dot{\zeta}} \right) , \qquad (59)$$

where 'dot' represents the derivative with respect to time t,  $n=1-3[1-\mu(1-\mu)/3]/2c^2$ , with c being the speed of light. In addition,  $U=U_c+U_r$  is the RR3BP potential function composed of the classical potential  $U_c=r^2/2+(1-\mu)/r_1+\mu/r_2$ , with  $r=\sqrt{\upsilon^2+\zeta^2}$ ,  $r_1=\sqrt{(\upsilon+\mu)^2+\zeta^2}$  and  $r_2=\sqrt{(\upsilon+\mu-1)^2+\zeta^2}$ . The relativistic corrections  $U_r$  are given by

$$U_{r} = -6r^{2}c^{2} \left[ 1 - \frac{1}{3}\mu(1 - \mu) \right] + \frac{c^{2}}{2} \left[ (\upsilon + \dot{\zeta})^{2} + (\zeta - \dot{\upsilon})^{2} \right]^{2}$$

$$+ 6c^{2} \left( \frac{1 - \mu}{r_{1}} + \frac{\mu}{r_{2}} \right) \left[ (\upsilon + \dot{\zeta})^{2} + (\zeta - \dot{\upsilon})^{2} \right] - 2c^{2} \left( \frac{1 - \mu}{r_{1}} + \frac{\mu}{r_{2}} \right)^{2}$$

$$- 2c^{2}\mu(1 - \mu) \left[ \frac{1}{r_{1}} + \left( \frac{1}{r_{1}} - \frac{1}{r_{2}} \right) (1 - 3\mu - 7\upsilon - 8\dot{\zeta}) + \zeta^{2} \left( \frac{\mu}{r_{1}^{3}} + \frac{1 - \mu}{r_{2}^{3}} \right) \right] .$$
 (60)

The libration points for the RR3BP are obtained by substituting  $\ddot{v} = \ddot{\zeta} = \dot{v} = \dot{\zeta} = 0$  in the above equations of motion [Contopoulos, 1976, Bhatnagar and Hallan, 1998]. These libration points are the relativistic counterparts of the Lagrange collinear  $L_1$ ,  $L_2$  and  $L_3$ , and triangular  $L_4$  and  $L_5$  points. Linear stability of the relativistic collinear points was investigated by Ragos et al. [2000] and Douskos and Perdios [2002], who showed that all these points were unstable, which is consistent with the results obtained for the non-relativistic collinear points.

In the work by Bhatnagar and Hallan [1998], linear stability of the relativistic triangular  $L_4$  and  $L_5$  points was studied and it was demonstrated that these points were unstable for the whole range  $0 \le \mu \le 0.5$ , despite the well-known fact that the non-relativistic  $L_4$  and  $L_5$  are stable for  $\mu < \mu_0$ , where  $\mu_0 = 0.038521$  is the Routh critical mass ratio (see Section 6.2). The problem was later revisited by Douskos and Perdios

[2002] and Ahmed et al. [2006], who found that the relativistic triangular points are linearly stable in the range of mass rations  $0 \le \mu < \mu_r$ , where  $\mu_r = \mu_0 - 17\sqrt{69}/486c^2$  [Douskos and Perdios, 2002], and  $\mu_r = 0.03840$  [Ahmed et al., 2006].

Among different possible applications of the RR3BP as discussed by Brumberg [1991] and Kopeikin et al. [2011], let us also mention the calculation of the advance of Mercury's perihelion made by Mandl and Dvorak [1984], the computation of chaotic and non-chaotic trajectories in the Earth-Moon orbital system by Wanex [2003], the relativistic corrections to the Sun-Jupiter libration points computed by Yamada and Asada [2010a], the analysis of stability of circular orbits in the Schwarzschild-de Sitter space-time performed by Palit et al. [2009], and the investigation of the GTR effects in a coplanar, non-resonant planetary systems made by Migaszewski and Goździewski [2009].

# 10. Summary and concluding remarks

We have reviewed the three-body problem in which three spherical (or point) masses interact with each other only through gravitational interactions described by Newton's theory of gravity, and no restrictions are imposed on the initial positions and velocities. We began with a historical overview of the problem, where a special emphasis was given to the difficulty of finding closed form solutions to the three-body problem due to its unpredictable behavior. We gave detailed descriptions of the *general* and restricted (circular and elliptic) three-body problems, and described different analytical and numerical methods used to find solutions, perform stability analyses, and search for periodic orbits and resonances. We also discussed some interesting applications to astronomical systems and spaceflights. In the previous section, we also presented the general and restricted relativistic three-body problem and discussed its astronomical applications.

The three-body problem described in this paper may be considered as classical (or standard) because the bodies were assumed to be spherical objects or points of given masses as described by either Newton's theory of gravity (Sections 1-8), or by Einstein's theory of gravity (GTR) in its post-Newtonian approximation (Section 9). However, there are studies of the effects of oblateness as well as Coriolis and centrifugal forces on the three-body problem [Abouelmagd et al., 2013], and the existence of the libration points in the restricted three-body problem with one or both primaries being oblate spheroids [Markellos et al., 1996], or both primaries being triaxial rigid bodies [Sharma et al., 2001]. Moreover, the existence of periodic orbits in the restricted three-body problem with one primary being an oblate body [Mittal et al., 1996], or with all objects being oblate spheroids and the primaries being radiation sources [Abouelmagd and El-Shaboury, 2012], have been investigated in the elliptic restricted three-body problem. Similarly, the effects of photogravitational forces in such systems have also been studied [Kumar and Ishwar, 2011, Singh and Umar, 2012]. Note that we cite only selected papers in which references to previously published papers on these

topics are given.

Also of interest is the so-called Chermnykh problem in which a point mass moves in the gravitational field produced by an uniformly rotating dumb-bell [Goździewski and Maciejewski, 1998, Goździewski, 2003]. This problem generalizes two classical problems of celestial mechanics, namely, the two-fixed center problem and the restricted three-body problem, with the former applied to the semi-classical quantization of the molecular ion of hydrogen [Goździewski and Maciejewski, 1998]; however, any applications of the three-body problem outside of celestial mechanics and dynamical astronomy are outside of the scope of this paper.

We would like to point out that our reference list is only a small subset of all papers and books published on the three-body problem. The choice of references was determined by what we considered to be the most relevant to the topics covered in this paper; however, we do respect the fact that other authors may have different opinions in this matter. Moreover, our reference list contains some papers with astronomical data where the results directly relate the applications of the general and restricted three-body problem to the Solar System, and newly discovered extrasolar planetary systems.

We hope that we presented a balanced view of the three-body problem, and that our choice of specific topics covered in this paper represents a good recognition of the previous and current research in the field. We also do sincerely hope that this review may serve as a guide to the major previous and current achievements in the field, and as an inspiration to scientists and students for opening new frontiers of research in this truly remarkable and very interesting problem.

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